THE DEVELOPMENTAL BASES
FOR EARLY CHILDHOOD
NUMBER AND OPERATIONS STANDARDS

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How important is it for early childhood and special educators to be knowledgeable about children’s mathematical teaching and learning? Current teaching practices and teacher-training efforts suggest that it is not a priority or even important. Consider, for instance, that at the University of Illinois at Urbana-Champaign (UIUC), elementary education majors are required to take two mathematics methods courses for a total of 6 credits (in addition to two mathematics content courses). In contrast, early childhood education majors take a single combination mathematics-science methods course. In effect, they take 1.5 credits of mathematics methods, less if science education faculty teach the course. Even more grievously, special education majors are not required to take even a single mathematics methods course. Unfortunately, an inadequate preparation for teaching young and special children mathematics is not unique to the early childhood and special education programs at UIUC.

The premise of this chapter is that early childhood and special education teachers need a powerful and practical framework for teaching young children and those with special needs. This framework should include a deep understanding of the following:

- the mathematics taught and how it relates to subsequent mathematics instruction (content);
- effective techniques for teaching mathematics (methods); and
- the development of children’s mathematical thinking and knowledge (mathematical psychology).

In this chapter, I will focus on the third component above, particularly number and operation developments for two reasons. One is that this is the area of my research specialization. The second is that number and operations are key elements of the core or foundation for all school mathematics (NCTM, 2000). In Part I of the chapter, I describe how psychologists’ view
of young children’s mathematical competence changed over the course of the twentieth century. I then make a case for an emerging view, one that can provide those responsible for early childhood and special education a general, but useful, developmental framework for making broad policy, curriculum, and instructional decisions. In Part II, I summarize some of what researchers have recently discovered about the development of young children’s number and arithmetic—knowledge that can provide early childhood and special education practitioners with the specific developmental framework necessary to make effective everyday educational decisions. In Part III, I draw some conclusions about the role of psychological research in changing early childhood mathematics instruction. In Part IV, I discuss why and how the early childhood standards should help early childhood and special education teachers construct a powerful developmental framework. Next, I outline how this could be accomplished in a practical manner. I end the chapter with a conclusion about the importance of ensuring a truly professional training for early childhood and special education practitioners—one that includes a sound developmental framework.

**Part I:**

**Changing Views of Young Children’s Numerical and Arithmetic Competencies**

Like a pendulum, the conventional wisdom about young children’s mathematical competence has swung from extremely pessimistic to extremely optimistic and is now beginning to swing back toward the middle. In this part of the chapter, I briefly describe the first pendulum shift (from extremely pessimistic to extremely optimistic view) and then detail the more recent shift (toward a middle ground). Consistent with this latest shift, I outline three plausible developmental transitions, developments that probably need to be a focus of early childhood standards and instruction.

**THE EARLIER PENDULUM SHIFT**

Over the course of the twentieth century, psychologists came to dramatically different conclusions about young children’s mathematical competence. For most of the century, scholars
held a pessimistic view and focused on what children can’t do. In the last quarter of the century, psychologists adopted an optimistic view and focused on what children can do (Gelman, 1979). For instance, whereas James (1890) described an infant’s perception of the world as “great, blooming, buzzing confusion,” Wynn (1998) noted that “findings over the past 20 years have shown infants are sensitive to number” and can even reason about or operate on very small ones (p. 5), well before they develop verbal-based counting competencies. Below, I outline these two diametric views of young children’s mathematical competencies.

**Pessimistic Views**

Although Edward L. Thorndike (1922) and Jean Piaget (1965) offered different views of mathematical teaching and learning, the work of both suggested that young children are mathematically incompetent.

**Thorndike’s Associative-Learning Theory.** Thorndike (1922) concluded that young children were so mathematically inept that “little is gained by [doing] arithmetic before grade 2, though there are many arithmetic facts that can [be memorized by rote] in grade 1” (p. 198). In his associative-learning view, the essence of mathematical learning is forming and strengthening bonds by rote. The heart of teaching is presenting material in a well-sequenced manner to take advantage of previously learned bonds and ensuring adequate practice to cement bonds. According to associative-learning theories, in general, children had to be rewarded (bribed) to learn mathematics, understanding was not central to learning useful mathematical skills, and students must be spoon-fed mathematics because they enter school uninformed and helpless. This view served as the rationale for the drill approach (Thorndike, 1922) and, years later, shaped the doctrine of direct instruction (Bereiter & Englemann, 1966). The lecture-and-drill method remains to this day the most widely used way to teach children of all ages (Ginsburg, Klein, & Starkey, 1998).

**Piaget’s Constructivist Theory.** In Piaget’s (1965) constructivist view, the essence of mathematical development is building cognitive structures—relational or meaningful knowledge and logical thinking. The heart of teaching is creating environments where children can explore
their world, challenge their current conceptions, and expand their ideas and thinking. In the constructivist view, young children have a natural curiosity. For example, they have an inherent desire to find patterns and resolve problems, the essence of mathematics. For Piaget, the construction of mathematical understanding was the heart of real development. For example, reflecting on the part-whole relations underlying addition, such as a whole is the sum of its parts and greater than any single part, advances mathematical thinking; the memorization of number facts by rote does not. In Piaget’s view, children actively construct their mathematical knowledge by interacting with their physical and social world.

For instance, by listening to their parents, older siblings, and peers, children detect counting patterns, devise counting rules, and sometimes over-apply these rules, rather than passively absorb (imitate) the counting-word sequence they hear. A clear indication of this is rule-governed counting errors such as fourteen, fifteen, sixteen, or twenty-eight, twenty-nine, twenty-ten (e.g., Baroody, 1987a; Ginsburg, 1977). (It is unlikely that parents, siblings, or preschool teachers model and reward fifteen or twenty-ten.)

There is now considerable evidence to support the constructivist views outlined above (see, e.g., Copley, 1999; Ginsburg et al., 1998). Indeed, there is even some evidence that suggests constructivist teaching and learning principles apply to special children, such as those with learning disabilities (e.g., Baroody, 1987a, 1996), mental retardation (e.g., Baroody, 1987a, 1999b), or behavioral disorders (e.g., Isenbarger & Baroody, in press).

Although Piaget’s (e.g., 1965) efforts were important in drawing researchers’ attention to young children’s informal mathematical knowledge and inspiring their efforts to examine it carefully (e.g., Gelman & Gallistel, 1978; Ginsburg, 1977), his work—in effect—reinforced Thorndike’s (1922) view that young children are not ready for number and arithmetic instruction (see, e.g., Kamii, 1985). He noted, for instance, that preschool children are not capable of abstract concepts or logical thinking. Thus, Piaget (1965) concluded that they are not capable of constructing a true concept of number or understanding of arithmetic.
Optimistic Views

As non-intuitive findings that challenge conventional wisdom are the most newsworthy (Rittle-Johnson & Siegler, 1998) and the most highly rewarded in science (Sagan, 1997), some psychologists reacted to the pessimistic view by looking for previously unsuspected mathematical strengths among preschoolers. Below I outline evidence of preschoolers’ numerical and arithmetic competencies and discuss some of the mechanisms proposed to account for these strengths.

Empirical Evidence Regarding Numerical Competencies. Using a habituation paradigm, researchers adduced evidence that infants can distinguish among collections of one, two, three, and (sometimes) four items but not among those involving four and five or more items (see, e.g., Ginsburg et al., 1998; Wynn, 1998, for detailed reviews). Indeed, such research indicated that infants are capable of numerical abstraction—can detect numerical equivalence across different contexts. Specifically, they can make equivalence judgments, even though the items comprising a habituation trial are each different, change for each habituation trial, and are different from the items comprising the comparison collections (Starkey, Spelke, & Gelman, 1990). Furthermore, infants appear capable of inter-modal transfer, such as equating a visual stimulus of three dots with an auditory stimulus of three drum beats (Starkey et al., 1990). While viewing displays of two or three objects, infants heard either two or three drum beats. Subsequently, they looked at collection with the same number of objects as drum beats.

Empirical Evidence Regarding Arithmetic Competence. With very small collections, even 4-month-old infants apparently expect addition to produce an increase in number and subtraction to result in a decrease in number (Wynn, 1992a, 1995). One “impossible-outcome” trial involves showing an infant an item, screening the item, and showing the infant another item. After the second item is moved behind the screen, the screen is removed to reveal one item (an “unexpected” outcome), two items (the “expected” outcome), or three items (an “unexpected” outcome). Infants look longer at the unexpected outcomes than they do the expected one (Wynn, 1992a).

Using a task invented by Brush (1978), Sophian and Adams (1987) found that toddlers
appeared to understand addition and subtraction. For instance, if items were simultaneously put into two containers one at a time (creating two equivalent and hidden collections) and then more items were added to one, participants indicated that this container had more items. Children also had some success on “critical” trials. On these trials, two inequivalent collections were first created and items were then added to the smaller collection, but not enough to make it equal to or larger than the other collection (e.g., putting five items in one container and two in another and adding one more item to the latter).

Using a simpler task, Starkey (1992) also examined preschoolers’ understanding of nonverbal addition and subtraction. Children from 18-months- to 5-years-old first watched balls being put into an opaque container. After watching either the addition or removal of one or more balls from the container, they then removed the balls from it one at a time. The number of times they reached into the container indicated their expectation of the resulting quantity. Most 2-year-olds seem to expect more balls after addition and fewer after subtraction. Often, the number of retrieval efforts made by children exactly matched the number of balls. However, success diminished as the initial quantity increased. For addition trials, accuracy dropped from 55% for $2 + 1$ to 15% for $3 + 1$. Overall, subtraction items were easier, but performance still declined from 91% for $2 - 1$ to 52% for $3 - 1$.

**Possible Mechanisms Underlying Early Competence.** The most pessimistic explanation for the infant number-discrimination and arithmetic-reasoning results—other than they are the product of wishful thinking (experimenter bias)—is that infants are not responding to numbers at all but to a perceptual cue. However, obvious perceptual cues, such as length, area, and density, have been discounted, because infant researchers typically controlled for them.

Subitizing—which can involve, but is not limited to, pattern recognition—has long been offered as explanation for pre-counting numerical competencies, in part, because it is typically limited to small numerosities (e.g., Klahr & Wallace, 1973; Mandler & Shebo, 1982). Starkey and Cooper (1995), for example, demonstrated that even 2-year-olds who cannot count can subitize, and their subitizing is not limited to objects arranged in a particular pattern.
Proponents of the extremely optimistic view have proposed a radically different mechanism to account for the apparent numerical competencies of pretoddlers. Gallistel and Gelman (1992) suggested that infants’ numerical competence derives from an accumulator mechanism originally proposed by Meck and Church (1983) to account for rats’ ability to discriminate number. As Figure 1 indicates, this mechanism—unlike other proposed nonverbal mechanisms for distinguishing among numerosities—entails a nonverbal counting process governed by the same principles that govern verbal-based counting (e.g., Gelman & Brenneman, 1994; Wynn 1996, 1998). In other words, infants are considered capable of abstract numerical representations based on an enumeration process, rather than a relatively concrete representation based on a low-level perceptual process (Wynn, 1996). This mechanism is further invoked to account for toddlers’ apparent success on nonverbal numerical-reasoning and computation tasks (e.g., Starkey, 1992; Wynn, 1995, 1998) and adults’ rapid and nonverbal apprehension of number (Whalen, Gallistel, & Gelman, 1999).

THE MOST RECENT PENDULUM SHIFT

At the end of the century, however, some researchers began questioning the optimistic view and focused on detailing more accurately what children can and cannot do. Below, I briefly discuss recent empirical evidence that challenges the extremely optimistic view. I then characterize the current state of understanding about pretoddlers’ knowledge of number and arithmetic. I then make a case for the mental model (Huttenlocher, Jordan, & Levine, 1994), a more balanced view of young children’s developing numerical and arithmetic competencies and one that includes three key developmental transitions.

Mounting Evidence Against the Extremely Optimistic View

Questions have been raised about the extremely optimistic view, because some of its supporting evidence has not been replicated, and earlier studies had methodological flaws that allow for plausible alternative explanations (see, e.g., Mix, Huttenlocher, & Levine, in press; Simon, 1997; Sophian, 1998). Consider seven examples.

Infant Attention to Number? Several recent studies contradict Wynn’s (1996) claim that
infants are sensitive to number rather than perceptual cues. Feigenson and Spelke (1998) found that after being shown one or two large objects, infants exhibited greater dishabituation to a change in size than a change in number. Clearfield and Mix (1999) argued that because size was not systematically varied, the significant (albeit smaller) dishabituation to number found by Feigenson and Spelke (1998) may also have been due to size. They tested this conjecture by systematically contrasting number with (the non-obvious perceptual cue) contour length (the total of the perimeters of individual objects in a collection). After being shown displays of two or three squares, 6- to 8-month-old infants dishabituated when shown the same number of squares with a different contour length but not when shown a different number of squares.

Nonverbal Counting by Infants? Uller, Carey, Huntley-Feener, and Klatt (1999) concluded that their results were inconsistent with the claim that infants use a nonverbal counting mechanism (Gallistel and Gelman, 1992; Wynn, 1996, 1998) and consistent with an “object-file” model. In their view, pre-toddlers encode numerosity by representing each item in a collection with a distinct symbol or marker (i.e., use an “object-file”). According to the accumulator model, infants should find the nonverbal addition problems involving a single update (e.g., $2 + 1$) as easy as those involving successive, multiple additions (e.g., $1 + 1 + 1$), because in the latter the infant is using a cardinal representation of first sum to determine the second (analogous to counting-on). According to the object-file model, a task that requires more updates should be more confusing to infants, because of the necessity to update object files with each transition. Uller et al. (1999) found that successive additions are more difficult with the same sum, a result that is inconsistent with the accumulator model (e.g., Gallistel & Gelman, 1992; Wynn, 1998) and consistent with Baillargeon, Miller, and Contantivo’s (1994) earlier results. (A possible difficulty with such research, however, is that the former are performed over a longer period of time, and attention may be a confounding factor.)

Infant Judgment of Equivalent Collections? Mix (1999b) argued that detecting changes in numerosity over different contexts—as done in infant habituation studies (e.g., Starkey et al., 1990)—does not involve an explicit or direct comparison of two collections. “When tasks re-
quiring direct comparisons are used, the evidence for early numerical abstraction is less compelling” Mix, 1999b, p. 272). In fact, success on such tasks emerges relatively late and in piecemeal fashion (Mix, 1999b; Mix, Huttenlocher, & Levine, 1996; Siegel, 1973; Zimiles, 1996).

**Infant Cross-Modal Transfer?** The evidence that infants are capable of cross-modal transfer and, thus, possess abstract numerical abilities is, in fact, mixed. In two studies (Mix, Levine, & Huttenlocher, 1997; Moore, Benenson, Reznick, Peterson, & Kagan, 1987), infants exhibited the opposite pattern found by Starkey et al. (1990). Furthermore, Mix et al. (1996) found that 4-year-olds, but not 3-year-olds, were able to detect a numerical correspondence between auditory and visual stimuli. Although possible, it does not seem likely that infants possess a numerical competence that 3-year-olds do not seem to have.

**Infants’ Nonverbal Arithmetic?** Sophian (1998) concluded that, although Wynn’s (1992a) findings suggesting infants recognize 1 + 1 = 2 and 2 – 1 = 1 “are intriguing, they do not conclusively demonstrate that infants have knowledge of arithmetic relations” (p. 32). Plausible alternative explanations can also account for (at least some of) the data (see, e.g., Simon, Hespos, & Rochat, 1995). For instance, Sophian (1998) noted that the infants’ responses in Wynn’s (1992) study might—like those in number discrimination studies—be based on same/different judgments. In the case of 1 + 1, an infant need only know that the result of adding an item to another is something other than a single item to attend more to the choice of two than to the choice of one.

**Toddlers’ Nonverbal Arithmetic?** Huttenlocher et al. (1994) concluded that neither the Sophian and Adams (1987) study nor the Starkey (1992) study provides convincing evidence that toddlers are capable of exact nonverbal calculation. Participants in the former tended to choose the collection that had been manipulated, even on the critical trials (where objects were added to a collection but it was still smaller than the other collection). As a result, they performed only slightly above what would be expected by chance or random choosing on these trials. Indeed, “Cooper (1984), using a similar method, concluded that 2-year-olds do not understand that the initial numerosity of a [collection] is important for predicting the effect of addi-
tion or subtraction of items to that [collection]” (Huttenlocher et al., 1994, p. 287). Huttenlocher et al. (1994) further noted that Starkey’s (1992) results may be confounded by touch cues and that a mean success rate of greater than 50% does not necessarily indicate that a majority of his participants were successful on the task. In contrast, when Huttenlocher and her colleagues (see, e.g., Jordan, Blantero, & Uberti, in press, and Mix et al., in press, for reviews) used a nonverbal addition and subtraction task without touch cues and an analysis of individual data (rather than group or averaged data) their participants younger than 3.5-years-old or so were not capable of nonverbal addition and subtraction.

**Young Children’s Rapid Acquisition of Counting Skills?** Some researchers have noted that even very young children seem attuned to number-relevant information, such as the number words, and acquire verbal-based counting skills very quickly (e.g., Gelman & Meck, 1986, 1992). Proponents of the extremely optimistic view have suggested that this focus on number-related information and rapid acquisition of counting skills by preschoolers is guided by the same mechanism and principles that underlie infants’ nonverbal counting. Put differently, the pre-existing principles that govern nonverbal counting enable 2- to 3-year-olds to learn the cardinal meaning of number words relatively easily.

A body of research suggests, however, that such a process may be more difficult and take longer than suggested by proponents of the extremely optimistic view (e.g., see Baroody, 1992b; Fuson, 1988, 1992, for reviews). Wynn (1990, 1992b), for instance, found that although 2- and 3-year-olds knew the cardinal meaning of the words *one* and *two* and that other number words referred to numerosities, they did not realize to which specific numerosities these other terms referred. Specifically, they could put out one or two items upon request (invariably doing so without counting) but could not do the same for larger numbers. More importantly, children took a surprisingly long time (about a year) to learn how the verbal counting system represents number (Wynn, 1992b). Specifically, participants only gradually and sequentially learned to count out two and three items, and only after this achievement did they discover a cardinal meaning of larger numbers (i.e., generally recognized that the counting sequence could be used to
create a particular number of items). Wynn (1990, 1992b) concluded that this delay might be
due to the fact that infants’ representation of number is different from that used by post-
toddlers and may not direct the latter’s construction of verbal counting knowledge.2

The State of Our Current Knowledge

Exactly how infants and toddlers perform as they do on nonverbal number discrimination
and arithmetic tasks is not clear. After a brief overview of some of the discrepancies, I offer
reasons why the mental model (Huttenlocher et al., 1994) may be the best basis for making
pedagogical decisions about mathematics instruction in early childhood.

Conflicting Claims. Consider two recent studies. Wynn and Bloom (1999) directly tested
the perceptual-cues hypothesis (e.g., Clearfield & Mix, 1999) by habituating infants to either
two moving groups of three objects each (2 x 3) or four moving groups of three objects each (4 x
3) and then presenting them with two types of targets: (a) two groups of four objects each (2 x
4) and (b) four groups of two objects each (4 x 2). Note that both targets have the same number
of objects, which presumably controls for a variety of correlated variables, including contour
length. The infants in Wynn and Bloom’s (1999) study tended to dishabituate on the 2 x 4 tar-
get if habituated in the 4 x 3 training and on the 4 x 2 target if habituated on the 2 x 3 training.
In brief, they appeared to discriminate on the basis of number. However, if the infants treated
each moving group of items as the unit, perceptual cues such as contour length would have been
based on the dimensions of the group not the items making up the group.

Xu and Spelke (2000) found that 6-month-old infants could distinguish between relatively
large collections that differed in size by a 2:1 ratio (e.g., 8 to 16) but not by a smaller ratio (e.g.,
8 to 12). They concluded that because of the size of the collections used, infants could be seeing
an object file. Furthermore, because the use of perceptual cues such as display size and contour
length could be discounted, Xu and Spelke (2000) concluded that infants must be discriminating
large numbers on the basis of a number representation and that the mechanism involved an ap-
proximate, not an exact, representation of numbers. Although these researchers did not explicit-
ly identify how this approximation mechanism worked, they seemed to favor an accumulator
Unfortunately, there are a number of methodological problems with the Xu and Spelke (2000) study (K. Mix, personal communication, April 6, 2000): (a) Habituation conditions had only the same average area, and it is not clear that infants would represent and habituate to this derived “invariant.” (b) It is not clear that all participants were habituated before testing. (c) The choice of dependant measure makes interpretation of the results difficult.

**A Case for the Mental Model.** Uller et al. (1999) note that “both overattribution and underattribution of cognitive resources to infants have serious theoretical consequences” (p. 3). Both can also have serious practical consequences. When considering the nature and content of an early childhood mathematics program, it is essential to have an accurate picture of young children’s cognitive capabilities. However, given the current state of confusion about this, there are several reasons why a more cautious interpretation of young children’s capabilities seems warranted.

1. Significant differences in infant studies do not necessarily indicate that they use an exact representation of numerosity (Huttenlocher et al., 1994). Statistically significant differences indicate only a tendency to, say, discriminate three from two, not a consistent ability to do so. An estimation mechanism such as using contour length could also account for such a tendency.

2. Mix (1999a) directly tested the predictions of the last four models listed in Figure 1. She tested 3- to 5-year-old children’s ability to make nonverbal equivalence judgments with two simultaneously-presented collections, two sequentially-presented collections, and two sequentially-presented events. The pattern of the participants’ performance better matched the predictions of an object-token model than the item-subitizing, item + event subitizing, or accumulator models.

3. Modest success on tasks involving explicit or direct comparisons of equivalence does not begin to occur before 3 years of age; strong evidence of success does begin to materialize before 3.5 years of age (Mix, 1999b; Siegel, 1973; Zimiles, 1966).
4. Huttenlocher et al. (1994) found that a majority of children did not succeed on even the simplest trials (1 + 1 and 2 + 1) of the nonverbal addition and subtraction task until 3 years of age. Children between 2.5 and 3 years of age were successful on such trials only about 25% of the time. For practical or pedagogical purposes, it seems reasonable to assume that determining exact answers to even the simplest addition and subtraction problems is not a given at the end of toddlerhood.

For the reasons above, it seems safer to assume that infants and toddlers cannot represent numerosities exactly and that this capability develops around 3 years of age for most children. Even if infants and toddlers have only a modest innate predisposition to attend to numbers and the capacity to estimate them, it might be enough—in most environments—to prompt the spontaneous development of number sense (cf. Dehaene, 1997). Specifically, it might be enough to promote a basic understanding of number, including ordinal relations, addition as increasing number, and subtraction as decreasing number. In other words, it would be sufficient, by and large, to account for the findings of extant research on the number and arithmetic abilities of infants and toddlers.

THREE KEY DEVELOPMENTAL TRANSITIONS

In this section, I briefly describe the three key developmental transitions suggested by the work of Huttenlocher and her colleagues (e.g., 1994). Although their mental model focuses on changes in the way number is represented, I explain how it is consistent or compatible with models that focus on how number is understood (e.g., Piaget, 1965; Resnick, 1992). I also note how the mental model is consistent with key developmental data. More specifically, I consider how the nonverbal numerical and arithmetic competence exhibited by infants may be qualitatively different from that exhibited by preschoolers—that is, what might account for the first key transition in these competencies. Next, I consider the second key transition from nonverbal number and arithmetic competencies to verbal ones. Finally, I touch on the third key transition from nonverbal and verbal number and arithmetic competencies to those involving written symbols.
Transition 1: The Development of Exact Nonverbal Numerical and Arithmetic Processes

Proponents of the mental model argue that the development of a numerical representation is more complicated than that suggested by the optimistic view (see Mix et al., in press). According to the accumulator model, for example, children—from the start—differentiate between discrete and continuous quantities and can represent small examples of the former exactly. In this view, conventional counting knowledge is presumed to build directly on this existing pre-counting knowledge. According to the mental model, children may not initially differentiate between discrete and continuous quantities, and both may be represented inexactly in terms of a perceptual cue, such as contour length. The evolution of object individualization provides a basis for precounters to construct an understanding of one-to-one correspondence, which, in turn, provides a conceptual basis for discrete number.

The mental model’s pre-Transition 1 and Transition 1 phases are consistent with Piaget’s (e.g., 1965; Piaget & Inhelder, 1969) view of number development. He too concluded that children’s earliest understanding of number was nonverbal, tied to perception, and imprecise. That is, Piaget (1965) argued that number development begins before children acquire language or other conventional knowledge, stems from (reflections on) perceptual cues or actions, and, thus, at first, is essentially an estimation process. He also argued that one-to-one correspondence (rather than verbal-based counting) is the psychogical basis for the construction of a number concept.

A pre-Transition 1 phase is also consistent with Resnick’s (1992) view that mathematical thinking begins with protoquantitive reasoning (Level 1 in Table 1): qualitative reasoning about amounts of physical materials before the development of verbal counting. At this level, “comparisons of amounts are made and inferences can be drawn about the effects of various changes . . . on amounts; but no numerical quantification is involved” (Resnick, 1992, p. 403; emphasis added).

The period before children engage in quantitative thinking about counted collections may actually involve multiple levels or sublevels of development (Levels 0, 1, 1A, 1B, and 1C in Table
Two of these five phases might exist before Transition 1—a transition Resnick (1992) did not explicitly address. Level 0 might involve pure qualitative reasoning—(e.g., recognizing that non-quantified wholes can consist of parts). Level 1 might involve qualitative reasoning about inexactly estimated quantities (e.g., that a whole is more than its parts). Three of the five phases might exist after Transition 1. Level 1A might involve qualitative reasoning about exact but nonverbally represented quantities (e.g., the parts $B$ and $B$ together make some larger whole). Level 1B might involve quantitative reasoning about exact but nonverbally represented quantities (e.g., the parts $B$ and $B$ together make the larger whole $B$). The fifth level, Level 1C, may mark the beginning of Transition 2. At this level, children associate number words with subitized quantities (the part $B$ or “two” and the part $B$ or “one” together make the larger whole $B$ or “three”).

What might account for Transition 1? One key factor was suggested earlier. Around 2, children may begin to construct an understanding of one-to-one correspondence, which provides a basis for identifying and representing discrete quantities. For instance, Mix (2000) noted that her 21-month-old son retrieved two dog treats for two pet dogs in another room while saying, in effect, This [one] is for [the name of the first dog], and this [one] is for [the name of the second dog]. Such intuitive, everyday one-to-one correspondences may help lay the groundwork for an informal understanding of numerical equivalence and number.

Another key factor underlying Transition 1 is the development of a symbolic-representation ability. Huttenlocher et al. (1994) hypothesized that children’s ability to create a mental model of numerosities and, thus, to represent them exactly should begin to develop at about 2, when they begin to exhibit a variety of symbolic activities, such as symbolic play (see also Piaget, 1951). Children between 2- and 3-years of age, for example, become capable of using a picture to understand the layout of a real room or inferring the location of a hidden toy in a room from a model of a miniature toy and room (DeLoache, 1987, 1991). In brief, unlike any other model listed in Figure 1, including other object-file models, it follows from the mental model that chil-
Children do not develop the ability to represent exact numerosities until after toddlerhood.

The results of the Mix (1999a) study, described in Point 2 of the previous section, support this position. Specifically, consistent with the mental model (Huttenlocher et al., 1994) and inconsistent with the object + event subitizing and the accumulator model, children were not successful in making equivalence judgments with sequentially-presented collections until 3.5 and with sequentially-presented events until even later.

Transition 2: The Development of Verbal-Based Numerical and Arithmetic Competencies

Children’s nonverbal numerical and arithmetical competencies probably provide a scaffold or basis for assimilating verbal-based numerical and arithmetical knowledge (Transition 2). Existing evidence indicates that young children are, in fact, more successful on nonverbal versions of arithmetic tasks than on verbally-presented story problems, as well as symbolic “number-fact” tasks (Huttenlocher et al., 1994; Jordan, Huttenlocher, & Levine, 1992, 1994; Levine, Jordan, & Huttenlocher, 1992; see Jordan et al., in press, for a detailed discussion).

Like Piaget (e.g., 1965), Resnick (personal communication, February 21, 1997) concluded that numerical and arithmetic knowledge and thinking “is reconstructed at each successive level, using both what [a] child knows from the previous level and new knowledge.” In the case of Transition 2, the previous knowledge is that which is constructed during Transition 1 (nonverbal numerical and arithmetic knowledge), and new knowledge is provided by verbal-based counting and computational experiences. In terms of Resnick’s (1992) model, Transition 2 permits the emergence first of (narrow) quantitative reasoning, then (broader) numerical reasoning, and ultimately, (general) abstract reasoning (again, see Table 1). The first involves reasoning about specific quantities—numbers tied to a particular and meaningful context (e.g., 3 candies and 2 more candies is the same as 2 candies and 3 more candies). Numerical reasoning entails reasoning about numbers in the abstract—without reference to a particular and meaningful context (e.g., three and two more equals two and three more). Abstract reasoning involves recognizing general principles that apply to any context or numbers (e.g., the principle of additive commutativity: the order in which any two numbers are added does not affect their sum).
Transition 3: The Development of Written Representations

In this third major transition, children assimilate written representations to nonverbal and verbal-based knowledge of number and arithmetic. Transition 3 begins as early as about 3-years of age and, typically, is gradual. For reviews of the literature on this topic, see Donlan (in press), Munn (1998), and Sinclair and Sinclair (1986).

Although Resnick (1992) does not explicitly discuss Transition 3, the development of written representations may facilitate numerical and abstract reasoning. By computing and recording the sums for the written expressions $5 + 3$ and $3 + 5$, for instance, children may recognize that addition is commutative (i.e., whether 3 is added to 5 or 5 is added to 3 or—at a more abstract level—whether $b$ is added to $a$ or $a$ is added to $b$, the result is the same).

Part II:
Recent Developmental Research

In this second part of the chapter, I summarize the developmental research regarding two really big ideas that are the basis for the Number and Operations Standard for Grades Pre-K—2 in chapter 4 of the *Principles and Standards for School Mathematics* (PSSM; NCTM, 2000):

- **Really Big Idea 1**: One of the most essential of human tools, numbers can play several roles, involve numerous relations, and can be represented in various ways.
- **Really Big Idea 2**: Numbers can be operated on (used to perform computations) in various interrelated ways to model a variety of real-world transformations or situations.

I then discuss the implications of these findings for the proposed early childhood mathematics standards (EC standards). More specifically, I address six key areas of early number and arithmetic development, namely, using numbers to quantify collections, using numbers to compare collections, adding and subtracting single-digit numbers, understanding part-whole relations, equal partitioning, and grouping and place value. These informal concepts form the
core of young children’s number sense and provide a key basis for understanding and assimilating school-taught mathematics. For each area/concept, I first note the “big idea” underlying it. I then summarize the recent research findings about how each of the basic competencies develop between toddlerhood and the third year of formal number and arithmetic instruction (i.e., second grade) and how each is related to the developmental transitions discussed in Part I. Finally, I comment on whether the number and operations standard (and other relevant standards) for grades Pre-K to 2 mentioned in Chapter 3 and delineated in Chapter 4 of the PSSM (NCTM, 2000) adequately emphasize crucial components of these six competencies and then make recommendations for the forthcoming EC standards.

UNDERSTANDING, REPRESENTING, AND USING CARDINAL NUMBERS

- **Big Idea 1.1:** Counting can be used to find out how many in a collection or to make a collection of a particular size.

“Historically, number has been a cornerstone of the entire mathematics curriculum” (NCTM, 2000, p. 32). Fostering an understanding of number [Really Big Idea 1 above] is one of the most important aspects of early childhood instruction. Indeed, “all the mathematics proposed for prekindergarten through grade 12 is strongly grounded in number . . . . Young children’s earliest reasoning is likely to be about number situations, and their first mathematical representation will probably be of numbers” (NCTM, 2000, p. 32).

An understanding of cardinal number (Big Idea 1.1 above) deepens gradually over the course of early childhood and includes using counting to specify the size of a collection of things (enumeration) or to create a given number of items (production). An understanding of cardinal number also includes an ability to identify equivalent collections (e.g., recognizing ✶✶, ✶✶✶, ✶✶✶, and ✶✶✶✶✶ all as pairs or two, recognizing ✶✶✶, ✶✶✶✶✶, and ✶✶✶✶✶ all as trios or three, and so forth). Furthermore, it entails recognizing and properly using both informal and formal representations of numbers.
Development

**Pre-Transition 1.** Possible relations among the types of (a) number representation, (b) subitizing, (c) production (ability to produce an equivalent collection), (d) reasoning, and (e) arithmetic competence are summarized in Figure 2. Note that, before Transition 1, Level 1 children’s representation of number and subitizing are assumed to be inexact processes (*inexact nonverbal subitizing*). Thus, they can only create an imprecise equivalent collection, either while a model collection is still visible (*inexact nonverbal matching*) or after it has been hidden (*inexact nonverbal production*). The evolution of object individualization may provide a basis for exact nonverbal matching—the ability to create a new collection that is in one-to-one correspondence with a model collection (Mix et al., in press). The stage is now set for Transition 1 and a significant leap forward in number ability. (The effects of number representation on reasoning and arithmetic competence illustrated in Figure 2 will be discussed later in the section titled *SINGLE-DIGIT ADDITION AND SUBTRACTION.*)

**Transition 1.** The development of an ability to mentally represent small collections exactly by means of a mental image, mental marker, or other means (identified as *exact nonverbal subitizing* in Figure 2 and Item NN1 in Table 2) permits *exact nonverbal production* of small collections—the ability to create a collection that matches (is equivalent to) their mental representation of a previously viewed collection (Level 1A or 1B in Figure 2; Items NN1a and NN1c in Table 2; Huttenlocher et al., 1994). Transition 1 should also permit children to nonverbally and reliably identify as equivalent or nonequivalent two small, static (simultaneously presented) collections, both consisting of identical elements (NN1b in Table 2; Mix, 1999a; Siegel, 1973) or highly similar elements, such as one collection of disks and one of dots (NN1d; Mix, 1999b).

What these results seem to indicate is that 3-year-olds have already developed a nonverbal representation of number. Whether this representation consists of a mental picture, mental markers (analogous to tallies), or something else is not entirely clear. Nor is it clear whether this representation is an exact one or an estimate. In any case, the key finding is that 3-year-olds already have a reasonably accurate way of representing and comparing small collections of like
items before they even learn to count.

**Transition 2.** The assimilation of verbal-based number representations to nonverbal representations of number (Item NN2 in Table 2) appears to account for two important extensions in children’s ability to make both nonverbal and verbal equivalence judgments, both of which are consistent with Resnick’s (1992) model of increasing generality of thinking. The first is the ability to compare collections of increasingly dissimilar elements (Mix, 1999b). Unlike pre-Transition 1 children, those older than 3.5 years or so can compare two homogeneous collections of *dissimilar* objects, such as a collection of shells and a collection of dots (Item NN2a in Table 2; Mix, 1999b). Between 4 and 4.5, children can compare heterogeneous collections of dissimilar items (Item NN2e; Mix, 1999b; cf. Gelman & Gallistel, 1978).

The second important extension is that after 3.5 years, children become capable of making non-static comparisons—that is those that occur over time (Mix, 1999a). This includes comparing a collection presented simultaneously with one in which the items are presented successively (Item NN2b in Table 2; Mix, 1999a; Mix, Huttenlocher, & Levine, 1996), comparing a (simultaneous) visual display and a sequential auditory display (NN2c; Mix et al., 1996), and a (simultaneous) visual display and a sequential event (NN2d; Mix, 1999a).

Between 3.5 and 4 years of age, children develop verbal and object-counting skills (see Items VC1 to VC5 and VN1 in Table 2), skills that provide them a more powerful tool for representing and comparing numbers. *Exact verbal subitizing* (recognizing and labeling with a number word) one or two items (Level 1C in Figure 2 and Item VC3a in Table 2) and later up to about four items (Item VC3c) may be a basis for the transition from nonverbal numbering skill (non-verbal subitizing; Level 1B in Figure 2 and Item NN1a in Table 2) to verbal-based numbering skill (*enumeration*; Level 2 in Figure 2 and, e.g., Items VC3d and VC3e; von Glasersfeld, 1982). Similarly, *exact quasi-verbal production* (the use of subitizing rather than counting to create a collection of one or two items upon request (Level 1C in Figure 2 and Item VC4a in Table 2) and later up to about four items (VC4b) may serve as a basis for the transition from exact nonverbal production (Level 1B in Figure 2 and Item NN1c in Table 2) to *verbal production* (Level 2 in Fig-
ure 2 and Items VC4c and VC4d in Table 2). Exact verbal production may develop in two distinct phases. In response to a verbal request (e.g., “Give me three candies”), a child may initially use a subitizing process to grab and put out the correct number of items (subitizing-based, exact verbal production; Level 2a in Figure 2) and only later use counting to do so (counting-based, exact verbal production; Level 2b in Figure 2; Wilkins & Baroody, 2000). The later competence depends on the development of a verbal-based cardinality concept, which is discussed next.

The development of object-counting ability in the preschool years, described in part in the previous paragraph, is marked by a growing or deepening understanding of cardinality. The construction of a cardinality-principle (Gelman & Gallistel, 1978) or a count-cardinal concept (Fuson, 1988, 1992) is an important first step in meaningful object counting (see Item VC3d in Table 2). The construction of its “inverse,” namely the cardinal-count concept (Item VC4c), appears to be a key prerequisite for counting-based, exact verbal production (Level 2b in Figure 2; VC4d in Table 2; Baroody, 1987a; Wilkins & Baroody, 2000). A counting-based cardinality concept is further deepened when children discover the identity-conservation or number-constancy principle—that a collection after it has been counted does not change its cardinal value (Item VC3f; e.g., Piaget, 1964)—and order-irrelevance principle—that the order in which the items of a collection are enumerated does not matter as long as each item is counted once and only once (Item VC3g, e.g., Baroody, 1992b, 1993).

Transition 2 also enhances children’s ability to think about numerical relations. In addition to aiding preschoolers in making comparisons of small collections with different elements (e.g., Item NN2e), across modalities (Item NN2c), or across time (Item NN2d), their counting-based representation enables them to compare collections with more than four items. Specifically, by counting and visually comparing small collections, children can recognize the same number-name principle: Two collections are equal if they share the same number name, despite differences in the physical appearance of the collection (see VN2a, Baroody, with Coslick, 1998). Because it is a general (abstract) principle, young children can use it to compare any size collection that they can count.
Construction of the same number-name principle appears to underlie the later development of a relatively sophisticated or advanced understanding of numerical equivalence, what Piaget (1965) called number conservation. Number conservation involves recognizing the numerical equivalence of two uncounted and non-subitizeable collections over time and despite appearances—that is, over a number-irrelevant physical transformation that results in a misleading perceptual cue (VN2b). Initially, children do not realize that if one collection put into one-to-one correspondence with another is then physically, but not numerically, changed (e.g., lengthened or shortened), then it is still equivalent to the other collection. Even counting the collections may not help. After children construct the same number-name principle and gain confidence in this counting-based knowledge, they can then apply it in relatively complicated contexts such as the number-conservation task. This is, once they trust that this principle is applicable, children can disregard misleading perceptual cues and “conserve” equivalence relations (e.g., recognize that two previously matched collections still have the same number, even though one is longer and looks like it has “more”). In time, children construct the following general (qualitative) principle: If nothing is added to or subtracted from two equivalent collections, then they continue to have the same number (despite appearances). This relatively abstract principle allows children to conserve number with logical certainty—that is, without counting or rematching the items in the collections (see Baroody, 1987a, for a review of the research supporting this position).

**Transition 3.** The third major transition in children’s number knowledge is learning how to use written symbols to represent numerical situations or meanings. Sinclair and Sinclair (1986) noted that there are two important aspects of this symbolic competence: (a) children’s understanding or ideas about written symbols and (b) their personal production of these symbols when needed. Both of these aspects involve knowledge of both form and function.

- **Knowledge of form.** Knowledge of form includes constructing a mental image of a symbol. This entails recognizing the component parts of a symbol (e.g., noticing that a 6 consists of a “stick” and a “loop”) and part-whole relations (e.g., knowing how the parts of a 6 fit together...
to make the whole, such as the loop is attached to the bottom of the stick). Knowledge of the component parts and part-whole relations enables children to distinguish one written symbol from another and, thus, to identify and to read them. For instance, although 6 and 9 share the same parts, children can distinguish between them because the loop of each is attached to the stick in two distinct ways: bottom versus top and right versus left. The fact that the numerals such as 2 and 5 or 6 and 9 share common parts helps explain why some children—particularly those with learning difficulties confuse these numerals.

Knowledge of form also involves constructing a motor plan for symbols, a step-by-step plan of execution for translating a mental image into motor actions (Goodnow & Levine, 1973). A motor plan—which specifies where to start, what direction to proceed, when to stop, how to change direction, and where to stop—is necessary for copying and writing numerals. The plan for the numeral 6, for instance, specifies start at the top right and draw a stick that slants to the left; then make a loop by first going right.

Even if a child’s mental image for a written symbol is complete or accurate, he or she will not be able to write it correctly and may repeatedly make the same mistake if his or her motor plan is incomplete or inaccurate. This can even happen when children have a model numeral in front of them (i.e., when “merely” copying a numeral). Reversal (i.e., writing a symbol backward) is often the result of an incomplete or inaccurate motor plan. Writing 6 backwards some of the time indicates that a child may be unsure whether to start on the (top) right and draw a slanted line toward the left or vice versa. A child who consistently writes a 6 backwards may well have an inaccurate motor plan, one that specifies “start on the (top) left and draw a slanted line toward right.” Children with learning difficulties, particularly, may have such writing problems (Baroody, 1987a, 1988a; Baroody & Kaufman, 1993). (For a detailed discussion of how instruction can help children construct an accurate mental image and motor plan for each numeral, see Baroody, with Coslick, 1998.)

- **Knowledge of functions.** Knowledge of function includes knowing the various meanings of a numeral (namely cardinal, measurement, nominal, and ordinal meanings). It also involves
knowing when and why written representations would be useful and how they can be used effectively.

Hughes’ (1986) study indicates “that children do have personally meaningful ways of writing quantity before they use the conventional notation” (Munn, 1998, p. 60). He found that 3- and 4-year-olds most often used idiosyncratic representations (interpretable only to the child), pictographic representations (drawing of objects involved), iconic representations (e.g., tally marks), or a combination of pictographic and iconic representations. However, Hughes’ tasks did not have a clear communicative purpose, and his design was cross-sectional (Munn, 1998).

Munn (1998) adduced evidence that the key developmental shift in children’s use of representations is not form (moving from concrete to abstract representations) but in how they use symbols (from pre-functional to functional). She used the following problem-solving task: The child put 1, 2, 3, or 4 items in each of four cans and was asked to label the cans with a representation (e.g., a tally for the can containing one item, two tallies for the can containing two items, and so forth). The tester then explained that she was going to add an item to one of the cans (the two can) and that the child had to figure out which can had the added item.

Children that Munn claimed had a “functional” use of symbols used the mismatch between the representation and the number of items in the “two” can to solve the problem. These children also could apparently count out a collection to a verbal request. Other children who did not use the representational mismatch to solve the problem were considered “pre-functional.” Both those who were functional and nonfunctional symbol users employed a variety of representations (e.g., some functional users used pictorial representations; others used tallies or iconic representations; and others used numerals or symbolic representations). For this reason, she concluded that development regarding function was more important than that regarding form.

Munn (1998) pointed out two limitations of previous symbol-use tasks. One was that they did not create a real need to use symbols to communicate. Another was that they did not adequately measure an understanding of symbolic use. In regard to the latter, she noted that tasks
in which a child recorded the value of a collection and was later asked to read the representation and to recreate the collection require only transferring some counting principles to written representations. She concluded that a task requiring “a child to discriminate between a symbol that correctly represented a quantity and one that did not would be a far better test of the child’s thinking” (p. 61). Her problem, described above, avoided both of these two difficulties. However, children could have a “functional” understanding of written symbols and might not think to apply it in the particular context of Munn’s problem. In brief, her task might underestimate functional use.

Munn (1998) noted that there was an association between the functional use of number notation and children’s counting-based production ability. She concluded that this result indicates “a link between symbolic function and children’s use of the counting system [—that possibly] functional symbols in themselves might help children to integrate counting conventions with number logic more quickly than would happen otherwise . . . . The evidence presented here . . . suggests pre-school children’s understanding of numeric notation does indeed have a bearing on their number understanding” (p. 67). Aside from the danger of inferring a causal relation from a correlation, her conclusion about this developmental relation may be questionable because, as argued in the previous paragraph, her symbol-use task may have underestimated children’s functional understanding of written symbols. In brief, the functional use of written symbols might occur earlier than Munn’s results indicate, and it may or may not be highly associated with production ability.

Implications for EC Standards

Transitions 1 and 2. The PSSM (NCTM, 2000) correctly suggests that a major goal of the early childhood education should be building on and extending children’s rich and varied intuitive and informal knowledge of number. A mention of infants’ ability to recognize and discriminate small numbers (pre-Transition 1) accurately implies that number development begins before children acquire conventional knowledge such as counting. Furthermore, the importance of counting experiences is justifiably emphasized with such comments as “counting is a founda-
tion for students’ early work with number” (p. 79). The importance of Transition 2 is clearly implied by statements such as children “connect number words [and] the quantities they represent” (p. 78) and “can associate number words with small collections of objects” (p. 79).

However, no mention is made of Transition 1—what may be a key basis of intuitive and informal knowledge. Although (a) estimation, (b) mental representation of number, and (c) automatic recognition of number (subitizing) are discussed, their mention seems to be in reference to verbal-based number skills, not their possible predecessors (non-verbal number skills).

a. The only direct mention of number estimation (estimating the size of a collection) suggests that children use benchmarks (a known smaller quantity) to gauge the size of a relatively large collection. Recognizing that a collection of 12 items is a little more than 10 by visually noticing two groups of five, for instance, is likely only after children can readily identify and verbally label collections of five items and efficiently determine the sum of 5 + 5.

b. The following statement about mentally representing number is misleading because of the research cited: “In these early years, students develop the ability to deal with numbers mentally and to think about numbers without having a physical model (Steffe and Cobb, 1998)” (p. 80). If the PSSM writers had intended this statement to refer to Transition 1, then they should have cited Huttenlocher and colleagues (e.g., 1994). This transition in number-representation ability comes well before the more abstract representational abilities (e.g., mentally solving relatively difficult missing-addend problems) discussed by Steffe and Cobb (1988).

c. A distinction needs to be made between nonverbal subitizing (forming an exact mental representation of a collection) and verbal subitizing (forming an exact representation of a collection, which is associated with a verbal counting label). The former may develop as a result of Transition 1, whereas the latter is a by-product of Transition 2 (counting experiences). As Point a above illustrates, the contexts in which rapid recognition of number is discussed in PSSM (NCTM, 2000) suggest it focuses on only verbal subitizing.
The general guidelines for promoting Transition 2 are crucial but do not provide the detailed direction teachers need to help young children achieve this transition. For example, no mention is made of integrating the critical components for enumerating collections: (a) generating a correct verbal number sequence, (b) creating a one-to-one correspondence between verbal numbers and items in a collection, and (c) keeping track of which items have been counted and which have not. Furthermore, there is no indication when children face the challenges of doing so. (Three-year-olds often have difficulty coordinating the first two components, particularly when beginning or ending a count; whereas, 4.5- to 5.5-year-olds have the most difficulty with the third component above; Fuson, 1988).

The EC standards should include clear and explicit expectations about Transition 1, particularly in regard to nonverbal estimation, number representation, and subitizing. These standards should further make clear the possible role of these competencies in Transition 2—the development of verbal-based number estimation, mental representation, and subitizing. Moreover, some mechanism or mechanisms must be found to help educate early childhood and special education teachers, curriculum developers, and so forth about the specifics underlying both Transitions 1 and 2.

**Transition 3.** In a clear reference to Transition 3, the *PSSM* (NCTM, 2000) includes the following expectation for pre-K to 2 students: “connect . . . numerals to the quantities they represent” (p. 78). It is further noted that “concrete models can help students . . . bring meaning to [their] use of written symbols” (p. 80).

The *PSSM* (NCTM, 2000) also includes the expectation that young children should “develop understanding of . . . the ordinal and cardinal numbers and their connections” (p. 78). It further implies that they should understand the measurement meaning of number. No mention, however, is made of the nominal meaning (using numbers as names, such as “Your bus is number 24”).

The *PSSM* (NCTM, 2000) includes the clear expectation that children initially be allowed to use their own written representations of number (see, e.g., Figure 4.30 on page 131 and the sec-
ond and third paragraphs on page 136). Their use of pictographic or iconic representations, at least, is consistent with existing research (e.g., Hughes, 1986) and makes sense. If children use idiosyncratic representations, which are uninterpretable by others and, perhaps, in some cases, even by themselves later, then they miss the point about why we use written representations for numbers. Such children need help appreciating the communicative function of written representations.

Furthermore, the PSSM (NCTM, 2000) is silent about symbol-form instruction. The issues of how to help children construct a mental image of numerals or a motor plan for these symbols is not mentioned. Given that written representations of numerals and other mathematical concepts are indispensable aspects of nearly all mathematics instruction and innumerable everyday activities, it is essential that early childhood standards include some discussion of this topic.

The EC standards, then, should include the expectation that young students understand all meanings of number, including the nominal meaning. They should also include the recommendation that children develop and use their own invented written representations of numbers before learning conventional symbols—with the caveat that they be helped to understand the communicative function of symbols. Additionally, mechanisms must be found to help practitioners understand the mechanics of helping children learn to read and write conventional symbols.

UNDERSTANDING, REPRESENTING, AND USING ORDINAL RELATIONS OR NUMBERS

- **Big Idea 1.2:** Two (or more) collections can be compared or ordered, and numbers are one useful tool for doing so.

Object counting or numbering skills (enumeration and production) involve a single collection. Comparing quantities (numerical relations) can involve two (or more) collections of distinct items (discrete quantities). (It can also involve two or more continuous quantities, such as length, area, or time—quantities not composed of distinct and, thus, countable parts. Comparing continuous quantities is an aspect of measurement.) Comparing collections is a basic survival skill and also provides a basis for measurement. (In effect, measurement is the repeated application of a unit to a continuous quantity to partition it into equal-size and, thus,
countable parts.)

**Development**

**Transition 1.** Children appear to develop an intuitive (nonverbal) sense of number order (up to three or four) somewhat later than they do that of cardinal number. Some research suggests that understanding an array of three is more than an array of two items, for instance, develops between 12 and 18 months (Cooper, 1984; Strauss & Curtis, 1984). Currently, little is known about the transition from an inexact to an exact ordinal sense of numerosity.

**Transition 2.** One important aspect of Transition 2 is learning relational terms such as *more* and *less*. Children about 2 years of age can reliably identify as *more* the larger of two collections, as long as the perceptual cue(s) for the difference are salient—that is, one row of items is clearly longer, denser, or covers more area than another (Item VN1 in Table 2). This appears to be the beginning of assimilating language-based number knowledge to their nonverbal knowledge of numerosity. Gauging which of two collections is *less* is more difficult and develops later because, in part, children rarely hear or use the term (e.g., Donaldson & Balfour, 1968; Kaliski, 1962; Weiner, 1974).

A second important aspect of Transition 2 is learning how the counting sequence can be used to compare collections. Similar to their construction of a concept of equivalence and cardinality, preschoolers can further discover the *larger-number principle* by counting and visually comparing two unequal collections. The later a number word appears in the counting sequence, the larger the collection it represents (e.g., *five* represents a larger collection than *four* because it follows *four* in the counting sequence; Schaeffer, Eggleston, & Scott, 1974; see Item VN2 in Table 2). Once children can automatically cite the number after another in the counting sequence (e.g., The number after *four* is *five*), they can use the larger-number principle to *mentally* compare two adjacent numbers (e.g., Who is older, someone 9 or someone 8?—the 9-year-old because 9 comes after 8; Item VN4 in Table 2). This relatively abstract number skill has many everyday applications and can be used for even huge number (1,000,129 is greater than 1,000,128 because, according to our counting rules, the former comes after the latter). Typically, children can
cite the number after another up to ten and can use this knowledge and the larger-number principle to mentally compare any two numbers up to five before they enter kindergarten (are 4.5 to 5.5 years of age; Item VN4a in Table 2). By the time they leave kindergarten (are 5.5- to 6-years-old), children typically can compare any two numbers at least up to ten (Item VN4b in Table 2).

A third important aspect of Transition 2 is learning and applying the ordinal-number sequence (‘First, second, third, . . . ‘). The process of learning the ordinal terms can be facilitated by noticing the parallels (and dissimilarities) with the counting (cardinal) sequence (e.g., sixth → six + th; seventh → seven + th). The key to understanding and applying ordinal terms is recognizing, at least implicitly, that they are relational terms—defined relative to a reference point. For example, the “first in line” can be defined by which direction a line of children is facing or its direction of movement, such as toward the classroom door. Unfortunately, this defining attribute of ordinal numbers is often not made clear to children.

**Transition 3.** By assimilating written numerals to their knowledge of the verbal counting sequence, children can quickly recognize that written numbers embody an ordinal relation and choose the larger of two numerals (within their known counting sequence; Item WN3 in Table 2). Likewise, by connecting written representations for ordinal numbers (first, second, third, . . . or 1st, 2nd, 3rd . . . ) or ordinal relations (< and >) to their existing (verbal-based) understanding of numerical relations, children can readily understand and use these formal terms. Learning difficulties arise when children have not had the opportunity to learn prerequisite skills and concepts or when written representations for ordinal numbers are not related to this knowledge.

**Implications for EC Standards**

As with cardinal-number development, no mention is made of Transition 1. General guidelines are provided for promoting Transitions 2 and 3 (e.g., “develop understanding of the relative position and magnitude of whole numbers and of ordinal . . . numbers and their connection [to cardinal numbers]”; NCTM, 2000, p. 78). Again, though, these guidelines are not sufficiently detailed to help teachers plan instruction. Consider, for instance, the critical but often
overlooked skill of comparing numbers. The *PSSM* (NCTM, 2000) clearly lays out the expectation that students need to “develop understanding of the relative position and magnitude of whole numbers” (p. 78) and that they should be encouraged to discover that “the next whole number in the counting sequence is . . . more than the number just named” (p. 79). However, there are few clues about the developmental progression and prerequisites necessary to make such a discovery. (As suggested in Table 2, the prerequisites for fine-number comparisons, VN4, are VN3, the larger-number principle, and VC4a, automatically citing the number after another). In brief, the same conclusions made about the EC standards regarding cardinal number apply here also.

**SINGLE-DIGIT ADDITION AND SUBTRACTION**

- **Big Idea 2.1:** A collection can be made larger by adding items to it and made smaller by taking some away from it.

An understanding of addition and subtraction (a key aspect of Really Big Idea 2 noted earlier), which includes Big Idea 2.1 above, is fundamental to success with school mathematics and everyday life and should be a core topic of any early childhood curriculum. Such curricula should also focus on helping children devise and share increasingly efficient strategies for generating sums and differences.

**Development**

Recent research indicates that children start constructing an understanding of these arithmetic operations long before school. Whether this starts in infancy (pre-Transition 1) as more optimistic scholars claim (e.g., Wynn, 1998) or not, an intuitive understanding of addition and subtraction (Item NA1 in Table 2) clearly seems to develop before children are capable of verbal-based arithmetic efforts (post-Transition 2), which, in turn, develops before written arithmetic competencies (post-Transition 3).

As with making number comparisons, then, children’s informal addition is initially relatively concrete (in the sense that they are working non-verbally with real collections or mental representation of them) and limited to small collections of four or less. Later, as they master and can
apply their counting skills, they extend their ability to engage in informal arithmetic both in terms of more abstract contexts (word problems and, even later, symbolic expressions such as 2 + 1 = ?) and more abstract numbers (namely, numbers greater than four).

**Pre-Transition 1.** If the mental model is correct, then Pre-Transition 1 children’s reasoning about collections may be sensible but also inexact. This qualitative reasoning would permit such children to estimate small sums or differences (Level 1 in Figure 2).

**Transition 1.** Transition 1 children develop the ability to solve simple nonverbal addition or subtraction problems (e.g., Huttenlocher et al., 1994). Such problems involve showing a child a small collection (1 to 4 items), covering it, adding or subtracting an item or items, and then asking the child to indicate the answer by producing a matching number of disks. For one item plus another item (“1 + 1”), for instance, a correct response would involve putting out two disks rather than, say, one disk or three disks. In the Huttenlocher and colleagues’ (1994) study, for example, most children who had recently turned 3-years-old could correctly solve problems involving “1 + 1” or “2 – 1” (that is, they could imagine adding one object to another or could mentally subtract one object from a collection of two objects). Most who were about to turn 4-years-old could solve “1 + 2,” “2 + 1,” “3 – 1,” “3 – 2” as well, and at least a quarter could also solve “1 + 3,” “2 + 2,” “3 + 1,” “4 + 1,” “4 – 1,” and “4 – 3.” Thus, by the age of 4, children can mentally add or subtract any small number of items.

How do children so young manage these feats of simple addition and subtraction? They apparently can reason about their mental representations of numbers. For “2 + 1,” for instance, they form a mental representation of the initial amount (before it is hidden from view), form a mental representation of the added amount (before it is hidden), and then can imagine the added amount added to the original amount to make the latter larger. In other words, they understand the most basic concept of addition—it is a transformation that makes a collection larger. Similarly, they understand the most basic concept of subtraction—it is a transformation that makes a collection smaller.

As Figure 2 illustrates, the Post-Transition 1 phase may consist of three subphases. Even
after children acquire the symbolic ability to represent collections exactly (Transition 1), their reasoning about the effects of addition and subtraction may remain sensible but inexact—result in estimated sums or differences—at least for a short while (Level 1A in Figure 2). In the next sublevel (Level 1B in Figure 2), children would (gradually) be able to determine exact sums and differences for increasingly difficult items. This might be followed by third exact quasi-verbal addition phase (Level 1C in Figure 2), in which children determine exact sums and differences through a nonverbal process and subsequently attach a verbal label to an answer.

**Transition 2.** Later—but typically before they receive formal arithmetic instruction in school—children can solve simple addition and subtraction word problems (e.g., Carpenter & Moser, 1982; Fuson, 1992; Huttenlocher et al., 1994), including those involving numbers larger than four. How do they manage this? Basically, children decipher the meaning of the story by relating it to their informal understanding of addition as a “make-larger” transformation or their informal understanding of subtraction as a “make-smaller” transformation (e.g., Baroody, with Coslick, 1998; Carpenter, Hiebert, & Moser, 1983). They then—at least initially—use objects (e.g., blocks, fingers, or tallies) to model the meaning (type of transformation) indicated by the word problem. Consider the following problem: *Rafella helped her mom decorate three cookies before lunch. After lunch, she helped decorate five more cookies. How many cookies did Rafella help decorate altogether?* Young children might model this problem by using a concrete counting-all procedure: counting out three items to represent the initial amount, counting out five more items to represent the added amount, and then counting all the items put out to determine the solution (Achievement 1 in Table 3).

Research further reveals that children invent increasingly sophisticated counting strategies to determine sums and differences (e.g., Baroody, 1987b; Carpenter & Moser, 1983; Resnick & Ford, 1981). Points 2 to 6 in Table 3 summarize the key achievements involved for addition. At some point, children abandon using objects and rely on verbal (abstract) counting procedures, which require a keeping-track process: keeping track of how far to count beyond the first addend (Achievement 4 in Table 3). To solve the problem above, for instance, they might...
count up to the number representing the initial amount ("1, 2, 3") and continue the count five more times to represent the amount added ("4 [is one more], 5 [is two more], 6 [is three more], 7 [is four more], 8 [is five more]—8 cookies altogether"); the keeping-track process is the italicized portion). One shortcut for this strategy, disregarding addend, can reduce the effort required to keep track (Achievement 5 in Table 3). For 3 + 5, for instance, counting the larger addend first reduces the keeping-track process from five steps (as shown in the previous example) to two steps ("1, 2, 3, 4, 5; 6 [is one more], 7 [is two more], 8 [is three more]—the sum is 8"). Another shortcut many children spontaneously invent is counting-on (Achievement 6 in Table 3): starting with the number representing the initial (or larger) amount, instead of counting from one. For 3 + 5, for example, this would involve starting with the cardinal value of five and counting three more times: "5; 6 (is one more), 7 (is two more), and 8 (is three more)—8 cookies altogether."

Recently, questions have been raised as to whether children invent counting strategies such as concrete counting-all or learn them from others (e.g., Cowan, in press). Siegler’s latest model of addition development (SCADS), for example, builds on the assumption that concrete counting-all is learned by rote (Crowley, Siegler, & Shrager, 1997; Shrager & Siegler, 1998). This is based, in part, on Siegler’s (1997) report that mothers of 3-year-olds rarely talk about the goals (meaning) of addition, but that their actions implicitly underscore the key elements of concrete counting-all. Fundamental shortcomings of SCADS include not accounting for the evidence of nonverbal addition competence (e.g., Huttenlocher et al., 1994), the process of assimilation (Piaget, 1964), or the evidence that some children do self-invent concrete counting-all (Baroody, Tiiliakainen, & Liao, in press). Put differently, it is silent on two key developmental transitions: how verbally-based arithmetic competencies build on nonverbal ones, and how written arithmetic competencies build on both of these informal aspects of arithmetic knowledge (see, e.g., Jordan et al., in press, and Mix et al., in press, for discussions of the first transition, and Donlan, in press, and Munn, 1998, for discussions of the second).

**Transition 3 and Post-Transition 3.** Research evidence makes clear that instruction needs to ensure that written arithmetic representations should be connected to children’s informal
arithmetic knowledge (e.g., Baroody, 1987a; Baroody, with Coslick, 1998; Ginsburg, 1977; Hughes, 1986).

Probably one of the greatest concerns to early childhood educators is “number-fact” mastery. How the basic number combinations such as $7 + 1 = 8$ and $4 \times 5 = 20$ are registered in, represented by, and retrieved from long-term memory (LTM) are still baffling issues. Almost two decades ago, Mark Ashcraft (1982) published an article in *Developmental Review* summarizing the state of the knowledge in the area. Specifically, he concluded that:

1. Addition number facts were organized in LTM in a manner analogous to the addition tables studied by school children;
2. Rules involving 0 or 1 (e.g., any number times 0 is 0) served merely as slow backup strategies in case fact retrieval failed.

Research indicates that both these assumptions are probably wrong (see, e.g., Baroody, 1985, 1994, for reviews of the literature). For example, a series of training experiments (Baroody, 1988b, 1989a, 1992a) demonstrated that rules involving 0 and 1 transferred to unpracticed addition combinations, allowing children to answer such combinations efficiently. In a more recent review of the literature, Ashcraft (1992) himself concluded that the consensus in the field was (a) his original table-analogy model was no longer viable and (b) combinations involving 0 or 1 might well be produced by fast rules.

Research further indicates that other relational knowledge may play a key role in both number-combination learning and representation (Baroody, 1999a; Baroody, Ginsburg, & Waxman, 1983). For instance, research indicates that knowledge of commutativity may affect how basic combinations are represented (Butterworth, Marschesini, & Girelli, in press; Rickard & Bourne, 1996; Rickard, Healy, & Bourne, 1994; Sokol, McCloskey, Cohen, & Aliminosa, 1991). In and of itself, then, practice, is not THE key factor determining what number combinations children remember (e.g., Baroody, 1988b, 1999a), as suggested by information-processing models (e.g., Siegler, 1988; Siegler & Shipley, 1995). Another important finding is that even adults use a variety of strategies to determine sums and differences efficiently (e.g., LeFevre, Sadesky, & Bisanz,
1996; LeFevre, Smith-Chant, Hiscock, Daley, & Morris, in press). In brief, new research suggests that internalizing the basic number combinations is not simply a matter of memorizing individual facts by rote but may also involve automatizing relational knowledge and that experts do not simply retrieve facts from LTM but may use a variety of automatic or near-automatic strategies, including rules and other nonretrieval strategies.

**Implications for EC Standards**

**Transitions 1 and 2.** The PSSM (NCTM, 2000) does not explicitly address the development of nonverbal addition and subtraction (Transition 1) or how this might provide a basis for verbal-counting-based arithmetic (Transition 2).

In regard to post-Transition 2 developments, general, but not specific, guidelines are provided. Children’s use of informal addition strategies is encouraged. For instance, it is explicitly noted that “they often solve addition and subtraction problems by counting concrete objects, and many . . . invent problem-solving strategies based on counting strategies” (NCTM, 2000, pp. 79-80). Other general guidelines explicitly, or at least implicitly, noted include “students should encounter a variety of meanings for addition ad subtraction” (p. 34; see, e.g., Baroody, with Coslick, 1998, for a taxonomy of operation meanings), children should be “encouraged to develop, record, explain, and critique [sic] one another’s strategies for solving computational problems” (p. 35; see also p. 84) and instruction and practice should be done in context (in a purposeful manner). However, practitioners are not provided specific guidelines. For example, although teachers are encouraged to foster the relatively sophisticated counting-on strategy (Achievement 6 in Table 3), no advice is provided on how this can be accomplished (for such advice, see Part III of this chapter).

The forthcoming EC standards should include expectations regarding the development of nonverbal addition and subtraction and how their verbal counterparts can build on this knowledge. Mechanisms must also be found to help teachers learn specific guidelines for fostering young children’s nonverbal and verbal addition and subtraction.

**Transition 3 and Post-Transition 3.** In regard to Transition 3, instruction should build on
and extend children’s informal knowledge by helping them (a) to “connect . . . formal expressions or equations such as $5 + 3$ [and] $5 + ? = 8$” to problem situations and their informal solutions for them and (b) to “relate symbolic expressions to various problems” (Baroody, with Coslick, 1998, pp. 5-11 and 5-12). The PSSM (NCTM, 2000) implicitly, if not explicitly, includes the first recommendation by underscoring the importance of solving problems and connecting various representations of problem strategies and solutions. It clearly and explicitly makes in the second recommendation above in several places (see, e.g., pages 34, 83, and 139).

The recommendations regarding mastery of basic number combinations made in Chapters 3 and 4 are not entirely consistent, which may reflect the conflicting views held by the PSSM writers on this surprisingly complicated issue. The expectations for Pre-K to 2 outlined in Chapter 4 include “develop fluency with basic number combinations for addition and subtraction” (NCTM, 2000, p. 78), where fluency is defined as “using efficient and accurate methods for computing). Not equating fluency with the retrieval of isolated facts memorized by rote but equating it with various speedy and reliable methods is clearly consistent with recent research that even adults do so to determine basic sums (e.g., LeFevre et al., 1996). This key point is further reinforced by substituting basic number combinations for the misleading term basic number facts.

Importantly, Chapter 4 of the PSSM (NCTM, 2000) includes the recommendation that “students should develop strategies for knowing basic number combinations that build on their thinking about, and understanding of, numbers” and that they be provided tasks that “help them develop the relationships within addition and subtraction” (p. 84). A search of PSSM (NCTM, 2000) uncovered two examples of these recommendations: discovery of (a) the number-after rule (“the next whole number in the counting sequence is one more than the number just named”; p. 79) and (b) the complementary relation between addition and subtraction (e.g., $5 - 3 = ?$ can be thought of as $3 + ? = 5$; p. 138).

In Chapter 3, references are also made to “knowing basic number combinations” (pp. 32 and 33), but this appears to imply a different meaning than that suggested by Chapter 4.
contrast to the later chapter where knowing the basic number combinations is, at least implicitly, equated with fluency, a distinction between the two terms appears to be made in the earlier chapter. After the former is identified as essential comes the statement “equally essential is computational fluency” (p. 32). This same apparent distinction appears on page 35: “By the end of grade 2, students should know the basic addition and subtraction combinations, should be fluent in adding two-digit numbers . . . (italics added).”

The proposed EC standards should include explicit expectations about how written addition and subtraction can be linked to children’s existing knowledge, including the recommendation that they be encouraged to represent as equations word problems and their informally determined solutions. Given the common misconceptions and confusion about the issue, these standards should also include the following explicit and clear-cut expectations:

1. Fluency with each family of number combinations should build on two requisite developmental phases: (a) counting-based strategies for determining sums and differences (e.g., counting-on) and (b) reasoning-based strategies for doing so (e.g., \(7 + 8 = 7 + 7 + 1 = 14 + 1 = 15\)). (A family of combinations consists of combinations that share a common pattern or relation. A combination may belong to more than one family).

2. To promote the second (reasoning-based) phase and lay the groundwork for the third (fluency), teachers should encourage children to look for patterns and relations and use them to devise, implement, and share reasoning (“thinking”) strategies.

3. To promote the third phase (fluency) and to minimize the amount of practice required to achieve it, practice should focus on helping students automatize reasoning (thinking) strategies, not memorizing individual facts by rote.

4. Practice should be done in a purposeful, meaningful, and—when possible—inquiry-based manner.

5. Fluency can embody a variety of strategies, including—but not limited to—the recall of (isolated) facts.

Furthermore, the general principles above should be accompanied by at least one example.
For instance, Principle 2 could be illustrated by the examples cited in the PSSM (NCTM, 2000; the number-after rule for \( n + 1 \) combinations or translating subtraction combinations into known complementary addition combinations) or any one of the examples listed in Box 5.6 on pages 5-31 and 5-32 in Baroody, with Coslick (1998). Principle 4 could be illustrated by cases where problems, games, and other activities can be used to provide purposeful, meaningful, and inquiry-based practice (see, e.g., Box 5.4 and Activity Files 5.6 to 5.8 on pp. 5-28 and 5-29 of Baroody, with Coslick, 1998).

**PART-WHOLE RELATIONS**

* Big Idea 1.3/2.2: A quantity (a whole) can consist of parts and can be “broken apart” (decomposed) into them, and the parts can be combined (composed) to form the whole.

An understanding of how a whole is related to its parts—what Piaget (1965) termed “additive composition”—includes recognizing that a whole is the sum of its parts (Part 1 + Part 2 = Whole) and that the whole is larger than any single part (Whole > Part 1 or Part 2). The construction of a part-whole concept is an enormously important achievement (e.g., Resnick & Ford, 1981). Some scholars consider it to be the basis for a deep understanding of number (Really big Idea 1) and arithmetic (Really Big Idea 2) and a key link between these two concepts (e.g., Piaget, 1965; Resnick, 1992).

A part-whole concept may be the foundation for understanding the following more advanced concepts of number: (a) place-value representation (e.g., the whole 23 can be decomposed into the parts 12 tens and 3 ones or 1 ten and 13 ones), (b) common fractions (in the representation \( \frac{a}{b} \), the numerator \( a \) indicates the number of equal-sized parts of a whole of interest, and the denominator \( b \) indicates the total number of equal parts into which the whole is subdivided), and (c) ratios—including probability (the probability of an outcome = its frequency/frequency of all outcomes = frequency of the part of interest/frequency of all parts or the whole).

A part-whole concept is assumed to underlie a more formal part-whole ("binary") meaning
of addition and subtraction (Resnick, 1992). Unlike children’s informal change add-to view of addition (embodied in Problem A below), part-whole situations do not involve a physical action that results in increasing an initial amount (see, e.g., Problem B below).

✿ **Problem A.** Arillo had three candies. His mom gave him two more. How many candies does Arillo have now?

✿ **Problem B.** Bree held three of her candies in her left hand and two in her right hand. How many candies did she have in all?

A part-whole concept is considered to be a conceptual basis for understanding and solving missing-addend word problems such as Problems C and D below and missing-addend equations such as \( ? + 3 = 5 \) and \( ? - 2 = 7 \) (Resnick, 1992; Riley, Greeno, & Heller, 1983).

✿ **Problem C.** Angie bought some candies. Her mother bought her three more candies. Now Angie has five candies. How many candies did Angie buy?

✿ **Problem D.** Blanca had some pennies. She lost two pennies playing. Now she has seven pennies. How many pennies did Blanca have before she started to play?

A part-whole concept may be the psychological basis for arithmetic concepts such as the principles of additive commutativity (Part 1 + Part 2 = Part 2 + Part 1) and associativity ([Part 1 + Part 2] + Part 3 = Part 1 + [Part 2 + Part 3]; Resnick, 1992; Riley et al., 1983). Furthermore, an understanding of part-whole relations may serve to connect the operations of addition and subtraction in the following three ways: the basic complement principle, Whole – Part 1 = ? ⚫ Part 1 + ? = Whole, where ⚫ means related), the advanced complement principle (Part 1 + Part 2 = Whole ⚫ Whole – Part 1 = Part 2 or Whole – Part 2 = Part 1), and the inverse principle (Part 1 + Part 2 – Part 2 = Part 1 or Part 1 – Part 2 + Part 2 = Part 1; e.g., Baroody, 1999a).

Finally, a part-whole concept may underlie an understanding of “number families” or the different-names-for-a-number concept (a number can be represented in various ways because a whole can be composed or decomposed in various ways) and is one key link between number and arithmetic.

The number represented by 5 can also be represented by, for example, 0 + 5, 1 + 4, 2 + 3,
Development

**Part-Whole Concept.** Using a matching task to eliminate the need for verbal responses, Boisvert, Standing, and Moller (1999) found that a majority of children as young as 2.5 years of age could correctly identify a composite figure (the whole) made up of conceptually different units (the parts). For instance, asked to find a “cat made of triangles,” participants more often pointed to the corresponding picture than they did to a picture of a cat, a picture of triangles, and a picture of a giraffe made of triangles (PW1 in Table 2). The results contradicted earlier evidence that preschoolers cannot pay attention to both the whole and its parts simultaneously (e.g., Elkind, Koegler, & Go, 1964).

The construction of a part-whole concept may begin with inexact non-verbal (pre-Transition 1) experiences, such as putting together interlocking blocks or pieces of playdough and taking them apart. This could lead to an intuitive understanding that a whole is larger than its composite parts. Transition 1 could result in a more precise understanding of additive composition, at least with quantities children could nonverbally subitize and mentally represent. Specifically, it may provide the basis for recognizing that one discrete quantity and another invariably make a particular total (e.g., 2 items and 1 more item always yield 3 items) and, even perhaps, that the order in which these particular discrete amounts are combined does not affect the total.

Irwin (1996) examined the following two key aspects of a protoquantitative part-whole concept identified by Resnick (1992) and found that the majority of children as young as 4-years-old understood both:

1. The **co-variation principle** entails recognizing the effects on an uncounted whole of adding items to or subtracting items from a part (If Part 1 + Part 2 = Whole, then [Part 1 + a number] + Part 2 = Whole + the number of [Part 1 – a number] + Part 2 = Whole – the number; see Item PW2 in Figure 2).

2. The **compensation principle** involves understanding the effect of taking an item from one part and adding an item to the other part (If Part 1 + Part 2 = Whole, then [Part 1 + a number]
Irwin’s (1996) participants were significantly less successful on corresponding tasks with counted wholes (Item PW3) and utterly unsuccessful on a symbolic version involving verbally-stated numbers (PW7). Her results, then, appear to be consistent with Resnick’s (1992) model.\textsuperscript{8}

**Class Inclusion.** One task Piaget (1965) used to study the development of part-whole knowledge was the “class-inclusion” task. This task entailed showing children two collections such as five roses and three daisies and asking them if there were more flowers or more roses. Children before about 7 years of age typically responded that there were more of the latter.

However, subsequent analyses and research strongly indicate that Piaget (1965) underestimated young children’s part-whole knowledge, in part, because the wording of his class-inclusion task was unfamiliar and confusing to them (Brainerd, 1978; Kohstamm, 1967; Markman, 1979; Trabasso, Isen, Dolecki, McLanahan, Riley, & Tucker, 1978; Winer, 1980). When collective terms such as *family* or *army* are used in class-inclusion questions, children as young as 4 are successful on the task (Item PW4b in Table 2; Fuson, Lyons, Pergament, Hall, & Kwon, 1988; Markman, 1973). Controlling for a variety of extraneous difficulties, Sophian and McCorgray (1994, Experiment 2) found that 5- and 6-year olds, but not 4-year-olds were successful on a class-inclusion task.

Sophian and McCorgray (1994, Experiment 1) and Sophian and Vong (1995), likewise, found that 5- and 6-year-olds, but not 4-year-olds, recognized that the starting amount (Part 1) in a missing-start change add-to problem had to be less than the numerical total of two numbers (the Whole). Irwin (1996) noted that this accomplishment apparently reflects the transition from protoquantitative knowledge about class-inclusion relations to quantitative knowledge about it (Item PW4c in Table 2). This research is discussed further in the next subsection.

**Missing-Addend Problems.** Young children’s inability to solve missing-addend word problems and equations has been taken as yet more evidence that they lack a part-whole concept (e.g., Riley et al., 1983). Some have interpreted such evidence as support for Piaget’s (1965) conjecture that the pace of cognitive development limits the mathematical concepts chil-
Children can and cannot learn and have concluded that instruction on missing addends is too difficult to be introduced in the early primary grades (Kamii, 1985).

The results of several recent studies suggest otherwise (e.g., Sophian & Vong, 1995). Sophian and McCorgray (1994, Experiment 1), for instance, gave 4-, 5-, and 6-year-olds problems like Problems C and D above. Problems were read to a participant and acted out using a stuffed bear and pictures of items. When reference was made to the initial unknown amount, the participant was shown a round box covered by an envelope. When reference was made to adding objects, a picture of their objects were shown to the child and then put in the envelope (out of sight). For problems involving subtraction, a picture of the objects taken was removed from the box, shown to the child, and then placed out of sight. When the result was mentioned, the participant was shown a picture of the corresponding items. Although 5- and 6-year-olds typically had great difficulty determining the exact answers of such problems, they at least gave answers that were in the right direction. For Problem C, for instance, children knew that the answer (a part) had to be less than five (the whole). For Problem D, for example, they recognized that the answer (the whole) had to be larger than seven (the larger of the two parts). These results suggest that 5- and 6-year-olds can reason (qualitatively) about missing-addend situations and, thus, have a basic understanding of part-whole relations (Item PW5a in Table 2).

**Related Number and Arithmetic Concepts.** The available evidence does not provide a clear indication of when children construct a more formal part-whole view of addition and subtraction (see Baroody, Wilkins, & Tiilikainen, in press, for a review). Moreover, efforts to trace the development of additive commutativity from a protoquantitative level to general abstract reasoning have, to date, not confirmed the progression of levels hypothesized by Resnick (1992; again see Baroody et al., in press, for a review). What is known is that, between 5- and 7-years of age, children discover that whether Part 2 is added to Part 1 or vice versa, the sum is the same—whether the task involves unknown quantities, known quantities, or symbolic expressions (Item PW6 in Table 2; Baroody, 1987b; Baroody & Gannon, 1984; Baroody et al., 1983; Bemejo & Rodriguez, 1993; Cowan & Renton, 1996; Sophian, Harley, & Martin, 1995). Fur-
thermore, children appear to recognize commutativity earlier when the task involves part-whole problems than when it entails change add-to problems (Wilkins, Baroody, & Tiilikainen, in press). The latter may be more difficult because change add-to problems imply adding in a particular order. Children who can overcome this implied order constraint (e.g., recognize that five and three more has the same sum as three and five more) have constructed a relatively deep understanding of additive commutativity.

Even the basic complementary relation between addition and subtraction (Whole – Part 1 = Part 2 \( \star \) Part 1 + Part 2 = Whole) appears to be far less salient to young children than is additive commutativity (Baroody, 1999a; Baroody et al., 1983). This principle may be a basis for a reasoning out difference (e.g., \( 5 - 3 = \) ? can be thought of as \( 3 + \) ? = 5), discussed earlier (see VA6 and VA7 in Table 2).

One way many pre-first-graders’ understanding of part-whole relations is incomplete is that they may not recognize that a quantity (a whole) can be decomposed or created in various ways (e.g., Baroody, with Coslick, 1998; Baratta-Lorton, 1976). Young children’s change add-to view of addition and change take-away view of subtraction may contribute to this gap in knowledge (Baroody, 1987a). For instance, a child may believe that five and two more is seven (Part 1 + Part 2 = a Whole) and not realize that four and three more can have the same outcome ([Part 1 – 1] + [Part 2 + 1] = the Whole). In effect, children may have to rediscover the compensation principle at the level of abstract number (Item PV5 in Table 2). This may occur about the age of 7 years, when children begin to compute and either mentally compare sums and differences or compare written equations (Item PV8).

**Implications for EC Standards**

In the *PSSM* draft (NCTM, 1998), part-whole relations are mentioned in two passages, the first implicitly and the second explicitly: (a) “develop an understanding of multiple relationships among whole numbers by . . . composing [and] decomposing number” (p. 109); (b) “Children gradually develop part-part-whole concepts. For example, in a situation where there are 4 red balls and 3 blue balls, three and four are parts and seven is the whole” (p. 111). Nothing is
explicitly said, however, about why a part-whole concept is important (e.g., providing a basis for assimilating missing-addend problems) or how a teacher might foster its development.

Point a above was retained in the PSSM (NCTM, 2000) in a revised form as one of the Pre-K to 2 expectations. Inexplicably, though, the direct mention of the part-whole concept (Point b above) was not. As was the case in the PSSM draft (NCTM, 1998), the PSSM (NCTM, 2000) does not include direct mention of how a part-whole concept underlies an understanding of key concepts and skills. The discussions of the concepts of additive commutativity and associativity, missing-addend addition, the relations between addition and subtraction, measurement, place value, and other names for a number concept (e.g., see pp. 33, 34, 80, and 82-84), for instance, are not tied to a part-whole understanding. A discussion of fractions only implicitly makes a connection to this key idea: “Young children can be encouraged . . . to see fractions as part of a unit whole or of a collection” (p. 33).

The EC standards should explicitly identify the part-whole concept as one of the “big ideas” that forms the nucleus of early mathematics instruction, one that holds this nucleus together as a coherent body of knowledge. It should further illustrate why it is important (see, e.g., Resnick, 1992) and how this key knowledge can be fostered (see, e.g., Chapter 6 in Baroody, with Coslick, 1998).

GROUPING AND PLACE VALUE

- Big Idea 1.4/2.3: Items can be grouped to make a larger unit and, in a written multidigit number, the value of a digit depends on its position because different digit positions indicate different units.

Children with only a counting-based concept of number do not think in terms of grouped items (larger units such as tens, hundreds, and so forth). As a result, they view a verbal number such as twenty-three or a multidigit written number such as 23 as a collection of 23 items, not as two groups of ten and three units. Unlike children who speak Asian languages (see, e.g., Miura & Okamoto, in press, for a recent review), English-speaking children initially do not realize that a digit’s position within a multidigit written number indicates its value (e.g., for 23, the 2 in the
tens place represents 2 tens or $2 \times 10$, not 2 ones or $2 \times 1$).

An understanding of grouping and place value (Big Idea 1.4/2.3 above) underlies the meaningful use of multidigit written numbers and renaming procedures (involving either whole numbers or decimals). Children who have not been given the opportunity to construct these concepts may have difficulty learning multidigit skills. The symptoms of such difficulties have been widely observed and include “writing numerals as they sound” (e.g., writing *twenty-three* as 203; e.g., Ginsburg, 1977), making “face-value errors” (e.g., interpreting 23 as two of something, such as two uncircled items, and three of something else, such as three circled items, e.g., Ross, 1989), and using “buggy algorithms” (e.g., subtracting the smaller digit from the larger regardless of position, as in $254 - 67 = 213$; Ashlock, 1976; Brown & Burton, 1978; Buswell, 1926).

A grouping concept, which includes other base systems (grouping by numbers other than 10), also has a variety of everyday applications. For example, money equivalents are based on groupings of five (5 pennies = 1 nickel, 5 nickels = 1 quarter, 5 dimes = 1 half dollar, and so forth) and ten (10 pennies = 1 dime, 10 dimes = 1 dollar, and so forth), electronic devices such as calculators or computers operate on a base 2 system, and produce retailers regularly use groups of 12 to quantify their purchases and sales (12 items = 1 dozen, 12 dozen = 1 gross, 12 grosses = 1 great gross; see Baroody, with Coslick, 1998, 2000 for examples and explanations).

**Development**

As suggested by the discussion below, the construction of a grouping concept and place-value concept may begin before entering school but, for English-speaking children, probably does not become well-defined without instruction. (English, unlike Asian languages, does not clearly underscore either concept; see, e.g., Fuson & Burghardt, in press; Miura & Okamoto, in press). The development of a grouping and a place-value concept is intertwined with each other and other concepts.

**Transitions 1 and 2.** The construction of a grouping concept may begin with inexact non-verbal (pre-Transition 1) experiences, such as seeing two rain drops merge to make a larger one.
After Transition 1 (in the exact, nonverbal phase), preschoolers may repeatedly add two blocks to a toy pickup truck to create a “full load” (a larger unit of blocks), which is transported to make a house (an even larger unit of blocks).

Transition 2 makes it more likely that children will construct a broader, more accurate, and more explicit understanding of grouping. With the advent of meaningful counting (enumeration and production), children can create equal groups of even larger numbers and use numbers to ensure their equality. Dealing with larger numbers, in turn, can create a real need for forming groups of groups (i.e., even larger units). For example, in keeping track of hundreds of points, it behooves children to make piles of 10 (treat 10 items as a single group of ten) and to group 10 groups of ten into a hundred. This may lead to recognizing “hierarchical grouping”—that each larger unit is composed in the same way (e.g., tens are composed of 10 ones; hundreds, of 10 tens; thousands, of 10 hundreds; and so forth). Children take another important step when they recognize that grouping can be reversed (e.g., that a ten can be decomposed into 10 ones; e.g., Cobb & Wheatley, 1988).

The construction of a place-value concept may begin in pre-Transition 2 form when a young child, for instance, plays target games. A child in this phase of development may know that hitting the target is good and that hitting the bulls-eye of the target is even better. Transition 2, especially, can result in associating a particular value with, say, hitting the target and hitting the bulls-eye.

**Transition 3.** The transition to written representations of numbers, particularly informal ones (*twelve* = 12), can further facilitate the development of a grouping concept. It can be especially important for the construction of place-value understanding. A step toward an understanding that the position of a digit in a multidigit numeral defines its value may be recognizing that 23 and 32 represent different amounts or that the latter is larger than the former (Donlan, in press).

**Implications for EC Standards**

Grouping and place-value concepts are emphasized in the *PSSM* (NCTM, 2000) and justi-
The importance of using various models to help children construct these concepts is mentioned in a number of places. For example, one of the expectations for grades pre-K—2 is that children “use multiple models to develop initial understandings of place value and the base-ten number system” (NCTM, 2000, p. 78). Also clearly emphasized is the need to develop flexibility in thinking about numbers. The example of students modeling twenty-five with 25 beans and two dimes and a nickel, for instance, could be interpreted as helping children see that multidigit numbers have both a counting-based meaning and a grouping-based meaning. Key connections between place-value/grouping concepts and counting and written multidigit numbers are also discussed.

There is little mention, though, about the earliest forms of grouping and place-value concepts (pre-Transition 1 and Transition 1 experiences) in the PSSM (NCTM, 2000). Also not discussed are the relative merits of different models that can provide bases for Transitions 2 and 3 (see Figure 3) or the pedagogical and practical benefits of introducing other bases (see chapter 6 in Baroody, with Coslick, 1998 and 2000, for a discussion of both of these issues). Although the PSSM (NCTM, 2000) includes the caution that using concrete materials, especially in a rote manner, does not ensure understanding (e.g., Baroody, 1989b; Clements & McMillen, 1996), it does not lay out adequate guidelines for using manipulative in a meaningful manner (see, e.g., Baroody, with Coslick, 1998; Fuson & Burghardt, in press; Miura & Okamoto, in press). The EC standards should address all the issues raised above.

**EQUAL PARTITIONING**

*Big Idea 1.5/2.4: A quantity (whole) can be partitioned (decomposed) into equal size pieces (parts).*

Like a part-whole concept, equal partitioning is another big idea in that it provides the conceptual basis for important aspects of number (Really Big Idea 1) and operations on numbers (Really Big Idea 2) and, thus, can further connect these major domains. More specifically, subdividing a collection or other quantities into equal-sized parts (equal partitioning) is the conceptual basis for measurement, fractions, and division. Below, I’ll consider the last first, because
an understanding of equal partitioning appears to develop first in this domain, and this understanding can then be used to understand the first two domains.

Development

Even before they learn to count, children may be interested in sharing or splitting up (more or less) equally small, discrete (and continuous) quantities. With the development of counting, children secure another method of ensuring or checking on equal shares, particularly for quantities larger than four items.

Division. Research has shown that many children of kindergarten age can respond appropriately to fair-sharing problems involving divvying-up situations, such as Problem E below:

**Problem E.** Three sisters Martha, Marta, and Marsha were given a plate of six cookies by their mom. If the three sisters shared the six cookies fairly, how many cookies would each sister get?

Some children solve this type of problem by using a dealing-out strategy: Count out objects to represent the amount; then deal out the cookies one at a time into piles; repeat the process until all the objects have been passed out; and then count the number of items in each pile to determine the solution (e.g., Davis & Pikethly, 1990; Hiebert & Tonnesen, 1978; Hunting & Davis, 1991; Miller, 1984). For Problem E above, this would entail counting out six items, dealing out one item at a time to each of three piles, repeating the dealing-out process, and finally counting the two items in one of the piles to determine the answer. Even the operation of division, then, can be introduced at a concrete level to children as early as kindergarten.

Fair-sharing problems can also involve measuring-out situations, as Problem F below illustrates:

**Problem F.** Jerry has 12 cookies. If he made shares of 3 cookies each, how many children could share the cookies fairly?

An understanding of measure-out division is important for (a) a more complete understanding of the operation and, in time, how division is related to a groups-of multiplication and (b) making sense of fraction and decimal division. Research suggests, though, many children are not familiar with and have more difficulty solving measure-out problems (Fischbein, Deri, Nello,
Because they are inclined to relate symbolic division to a divvy-up meaning exclusively, students are puzzled by expressions such as $\frac{1}{2} \div \frac{1}{8}$ or $0.75 \div 0.25$ (Baroody, with Coslick, 1998).

**Measurement.** A deep understanding of what it means to measure begins with recognizing the distinction between discrete quantities and continuous quantities. The former are made up of distinct items that can’t be subdivided (don’t have “in-betweens”). To quantify or compare such quantities, we can count them. Continuous quantities—such as length, time, the amount of mathematical aptitude or anxiety—are not composed of clearly distinct parts and can be subdivided (have in-betweens). To quantify or compare such quantities, we can measure them. Measuring a continuous quantity entails subdividing it into equal-size parts, or *units*. Unfortunately, instruction often focuses on how to measure, not on what it means to measure (e.g., Baroody, with Coslick, 1998). Research shows, not surprisingly, that children have little understanding of measurement and often incorrectly use or interpret measurement tools (e.g., Bright & Hoeffner, 1993; Lindquist & Kouba, 1989).

**Fractions.** Equal partitioning of both discrete and continuous quantities to solve fair-sharing problems can provide a key informal basis for understanding fractions (e.g., Baroody, with Coslick, 1998; Hunting & Davis, 1991; Mack, 1990, 1993; Streefland, 1993). For instance, trying to share three cookies between two children can give rise to dividing each of the cookies into two equal-size pieces (halves) and giving each child a half of each cookie, or one whole cookie and half another ($\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$ or $1\frac{1}{2}$). Some children might solve the same problem by giving each child a whole cookie and half of the third cookie ($1\frac{1}{2}$). A class discussion could make explicit the following fundamental fraction concepts: (a) the shares must be fair or equal in size (fractions involve a special situation in which all the parts of a whole are equal in size), (b) *three halves* literally means *three one-halves* (fractions embody multiplicative reasoning; e.g., $\frac{3}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3 \times \frac{1}{2}$), and (c) *three halves and one and a half* represent the same amount (a fractional amount can have different names—which we call *fractional equivalents*). Unfortunately,
formal instruction on fractions typically moves too quickly with abstract symbols and, thus, does not provide an adequate conceptual basis for understanding fractions (e.g., the idea that parts must be equal; e.g., Behr, Harel, Post, & Lesh, 1992).

Fraction addition and subtraction may not deserve emphasis in early childhood education. However, as with an understanding of whole numbers (Piaget, 1965), a relatively complete and accurate understanding of fractions depends, in part, on understanding the role of additive composition. For instance, an accurate understanding of one-half includes recognizing that a half is less than a whole and that two halves ("1/2 + 1/2") make a whole.

Perhaps most surprising of all is the research indicating that preschoolers can understand simple fraction addition and subtraction. Mix, Levine, and Hutlenlocher (1999) presented 3-, 4-, and 5-year-olds nonverbal problems that involved, for instance, first showing half of a circular sponge and then putting it behind a screen, next showing half of another circular sponge and then putting it behind the screen also, and finally presenting four choices (e.g., one-quarter of a sponge, one-half of a sponge, three-quarters of a sponge, and a whole sponge) and asking which was hidden by the screen. The 3-year-olds were correct only 25% of the time (i.e., responded at a chance level—no better than could be expected by random guessing). The 4- and 5-years olds, though, responded at an above chance level. For instance, over half were correct on trials involving "1/4 + 1/2," "1/4 + 3/4," "1/2 − 1/4," and "1 − 1/4."

In his discussion of my paper presented at the Conference on Standards for Preschool and Kindergarten Mathematics Education (Baroody, 2000b), Kevin Miller correctly cautioned that Mix and her colleagues’ (1999) evidence does not necessarily mean that their participants understood fractions (e.g., as a part of a whole subdivided into equal-size parts), let alone fraction addition and subtraction. He noted that children could be responding to perceptual cues such as area. Clearly, further research is needed to assess whether most or even some 5-year-olds understand fraction addition and subtraction.
Implications for EC Standards

Division. The PSSM (NCTM, 2000) includes explicit mention of division and its conceptual basis (equal partitioning): “Understand situations that entail . . . divisions, such as equal groupings of objects and sharing equally” (p. 78). This document also clearly indicates that fair-sharing problems can provide a useful basis for developing these ideas: “Division can begin to have meaning for students in prekindergarten through grade 2 as they solve problems that arise in their environment, such as how to share a bag of raisins fairly among four people” (p. 34). The PSSM (NCTM, 2000) does not distinguish between divvy-up fair-sharing problems (like the one in the previous sentence or Problem E above) and measure-out fair-sharing problems (such as Problem F above).

The proposed EC standards should not only recommend the use of fair-sharing, in general, but the use of such problems involving both divvy-up and measuring-out situations. For a detailed discussion of the distinction between these two types of division problems and how a teacher can use everyday situations to introduce each, see pages 5-17 to 5-24, particularly Figure 5.4 on page 5-20 and activity 5.5 on page 5-21 in Baroody, with Coslick (1998).

Measurement. Measurement has its own standard and is discussed in detail in the PSSM (NCTM, 2000). Although the connection between measurement and equal partitioning is suggested, how the latter motivates the former is not clearly laid out. Consider the following passages: (a) “Measurement . . . bridges two main areas of school mathematics—geometry and number” (p. 103). (b) “Scales permit students to assign numerical values to the weights of objects (as rulers allow them to assign numerical values for linear measures)” (p. 104), and (c) In the measurement process, one must “choose a unit, compare the unit to the object, and report the number of units. The number of units can be determined by iterating the units (repeatedly laying the units against the object) and counting the iterations” (p. 105).

The connection between equal partitioning and measurement explains how the latter provides a key bridge between geometry and number. Space can be conceptualized in terms of continuous quantities such as length, area, and volume. Measurement, in effect, translates a con-
tinuous quantity into a discrete quantity—a quantity that is made up of distinct and, hence, countable items (Baroody, with Coslick, 1998). This is done, in effect, by equally partitioning a quantity (whole). This explains why Point b above is important and why we use the measurement process described in Point c.

The proposed EC standards should underscore how equal partitioning and measurement are connected (e.g., how measurement serves as a bridge between geometry and numbers, namely, why measuring is used to assign numerical values and why measuring involves iterating units). This should help make the discussion of the measurement standard and the examples used more meaningful to readers, providing them a clearer vision of how to teach young children in a meaningful fashion.

**Fractions.** The PSSM (NCTM, 2000) does include specific mention of fractions. For instance, page 33 includes the statement: “Beyond understanding whole numbers, young children can be encouraged to understand and represent commonly used fractions in context, such as \( \frac{1}{2} \) of a cookie or \( \frac{1}{8} \) of a pizza, and to see fractions as part of a unit whole or of a collection.” The list of expectations for Pre-K to 2 regarding number and operations on page 78 echoes the first portion of the previous statement (“understand and represent commonly used fractions, such as \( \frac{1}{4}, \frac{1}{3}, \text{ and } \frac{1}{2} \)”). Unfortunately, this key goal is not clearly related to the foundational concept of equal partitioning. The proposed EC standards should not only include the expectations about fractions outlined above but explicitly link this expectation to providing young children with equal-partitioning experience with discrete quantities first and then with continuous quantities. The former involves using a single divvy-up strategy for all problems, whereas the latter entails using different strategies for different problems (e.g., subdividing a cookie between two people requires finding a midpoint; subdividing it among three does not.)

For Pre-K to 2, the PSSM (NCTM, 1998) mentions operations on whole numbers only. Gaining understanding of fraction relations, including addition of fractions, is a goal for grades 3 to 5 (see, e.g., PSSM, page 33). Given that young children may be successful on nonverbal
fraction addition and subtraction tasks (Mix et al., 1999), it may make sense that—at the very least—early childhood instruction should involve them in qualitative reasoning about such operations using concrete models and, later, fraction words and written representations. Even if the 5-year-old participants in the Mix et al. (1999) were merely responding to area, this may provide a basis for qualitative reasoning about the effects of adding and subtracting fractions (halves at least). Along with equal-partitioning experience, this may provide a basis for an understanding of commonly used fractions such as $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{1}{2}$, including the recognition that two one-fourths (“$\frac{1}{4} + \frac{1}{4}$”) or two one-thirds (“$\frac{1}{3} + \frac{1}{3}$”) is less than a whole, and three halves is greater than a whole.

Part III:

The Roles of Psychological Research in Improving Early Childhood Mathematics Instruction

Psychological theory and research can play at least two important roles in promoting mathematics education reform: (a) changing beliefs about the learning and teaching of young children and (b) providing a powerful developmental framework for developing and guiding better instructional practices. In this part of the chapter, I note how recent psychological research has helped foster beliefs that early childhood mathematics instruction is possible and that a “child-centered” approach is not only feasible, but desirable. I then delineate the reasons why early childhood and special education teachers need a powerful developmental framework. Next, I describe an example of how recent psychological research can help mold beliefs about how young children are taught mathematics and why knowledge of such research is necessary to teach early childhood mathematics effectively.

FOSTERING BELIEFS ABOUT THE POSSIBILITY AND NATURE OF EARLY CHILDHOOD MATHEMATICS LEARNING

In this section, I discuss (a) the impact psychological research has already had on potential educational leaders’ beliefs about young children’s mathematical potential and how best to fos-
ter this potential and (b) some impediments to reaching a wider audience.

**Recognition of Young Children’s Potential**

**Recognition by Educational Leaders.** As illustrated in Parts I and II of this paper, recent research indicates that preschoolers do have impressive informal mathematical strengths in a variety of areas. In particular, it appears that young children—despite important limitations—are capable of understanding much more about number and arithmetic than previously or commonly thought possible. In her comments on the NCTM symposium “Linking Research and New Early Childhood Mathematics Standards (4/11/00), Maggie Myers correctly noted that this psychological research has already had an impact on educational, governmental, and industrial leaders, namely promoting the belief that mathematics development should be an important component of early childhood education and prompting efforts to create standards to guide this development. For example, on page 79 of Chapter 4 (“Standards for Grades Pre-K—2”), the following point is emphasized: “Teachers should not underestimate what young students can learn” (NCTM, 2000). In brief, psychological research has created a climate where discussing and developing early childhood mathematics standards is now possible.

**The Challenge Remaining.** What remains is, perhaps, a greater challenge—changing the belief of teachers, school administrators, and the public at large about mathematics learning in early childhood. This task is made more difficult because elementary education majors, overall, have relatively high levels of mathematics anxiety and mathematics avoidance (Hembree, 1990) and those of early childhood and special educators may be particularly high (see, e.g., Ashcraft, Kirk, & Hopko, 1998, for a review of the literature). The difficulty of the task is compounded by the largely negative attitude toward mathematics by the general public (e.g., Ashcraft et al., 1998; McLeod, 1992). Nevertheless, it is essential that early childhood and special education teachers and their supporting cast (which should include administrators and parents) need to be informed about young children’s mathematical capabilities.
Recognition of How Best to Foster Young Children’s Potential

It is important not only to change teachers’, other educators’, and the public’s view of young children’s mathematical potential, but to change their views about how it can be fostered. Below, I briefly summarize four different views of mathematics instruction and then discuss the challenge of changing the conventional or traditional view of teaching.

Four Views of Mathematics Teaching. Table 4 summarizes four qualitatively different approaches to instruction. The skills approach is the traditional approach, which Brownell (1935) labeled “drill theory.” The conceptual approach is essentially what Brownell (1935) called “meaning theory” and embodied his ideas for reforming the traditional skills approach. The problem-solving approach is basically what Brownell (1935) called “incidental-learning theory,” which was embodied in John Dewey’s early progressive-education movement and later in “Free or Open Schools” (e.g., Silberman, 1973) and some Piagetian (radical constructivists’) curricula (e.g., Furth & Wachs, 1974; Neill, 1960). The investigative approach, in effect, is a composite of what Brownell (1935) called the meaning and incidental-learning theories. This purposeful, meaningful, and inquiry-based approach embodies the NCTM (1989, 1991, 2000) Standards and “developmentally appropriate practices” (Bredekamp & Copple, 1997). (See, for example, Baroody, with Coslick, 1998, and Baroody et al., in press, for a detailed comparison of these four approaches.)

Both psychological and educational research and practical considerations suggest that the investigative approach is the best bet for promoting mathematical understanding (conceptual learning) and thinking (e.g., problem-solving and reasoning), as well as mastery of basic skills (Baroody, with Coslick, 1998; Baroody et al., in press). Furthermore, although more research is needed to settle the issue, there is some reason to believe that even children with severe learning difficulties might benefit from the investigative approach (see Baroody, 1996, 1999b, for reviews of the literature).

The Challenge. As Ginsburg et al. (1998) noted, psychological research has had and can have an important impact on educators’ and the public’s view of how mathematics should be
taught. Unfortunately, its impact on educational practice has not, to date, been entirely positive. Indeed, Ginsburg et al. (1998) argued that although William Brownell (1935) won the hearts and minds of mathematics educators with his meaning theory, Thorndike (1922) won the hearts and minds of practitioners with his drill theory. Many teachers and other adults cling to the view that teaching involves talking and learning involves imitating or practicing facts, definitions, procedures, and formulas until they are memorized by rote—in spite of their own dismal experiences with traditional drill-based instructional practices. In other words, many teachers and other adults believe that mathematics should be taught the way they were taught, despite the fact that they found it unappealing, anxiety-provoking, and/or largely unhelpful. In brief, the conventional view of mathematics is the skills approach, not the conceptual approach, let alone the more effective but more complicated standards-based investigative approach. The challenge remaining is convincing teachers and the public at large that the standards-based investigative approach is more effective than the traditional skills approach, or even the conceptual approach.

REASONS TEACHERS NEED A DEEP UNDERSTANDING OF MATHEMATICAL PSYCHOLOGY

Changing the beliefs of teachers (e.g., convincing them of the merits of a child-centered approach), however, is not enough. Indeed, many early childhood educators already believe in such an approach. Early childhood and special education teachers need a powerful and practical framework in order to make the innumerable decisions needed to implement the investigative approach effectively. Below, I outline the general rationale for why teachers need an extensive knowledge base, one that includes a deep understanding of mathematical psychology. I then outline three important ways this developmental framework can help educators. Next, I summarize some conclusions about the role of psychological knowledge in improving mathematics instruction.
General Reasons Why Early Childhood Educators Need a Powerful Developmental Framework

John Dewey (1963), the father of the progressive-education movement, recognized that his early efforts to implement a child-centered approach were not successful and concluded that simply providing children experiences in the form of free play or unstructured discovery learning (the problem-solving or incidental approach) did not ensure learning. He came to the following conclusions:

1. Educational reform cannot simply be a knee-jerk reaction to traditional instruction (a skills approach). That is, new teaching methods cannot be substituted for traditional methods simply because they are different from the latter. New teaching approaches, methods, or tools must have their own (theoretical, empirical, and practical) justification. The PSSM (NCTM, 2000), particularly Chapters 1 (“A Vision for School Mathematics”) and 2 “Principles for School Mathematics”—along with previous NCTM (1989, 1991, 1995) standards documents—provides a well articulated justification for current reform efforts.

2. Teachers must strive to provide **educative experiences** (experiences that lead to learning or a basis for later learning), not **mis-educative experiences** (experiences for experience’s sake and that may impede development). This sentiment is reflected in the following statement in the PSSM (NCTM, 2000): “High-quality learning results from formal and informal experiences during the preschool years. ‘Informal’ does not mean unplanned or haphazard” (p. 75). It is further reflected in the “curriculum principle”: “A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across the grades” (NCTM, 2000, p. 14).

3. Educative experiences result “from an interaction of external factors, such as the nature of the subject matter and teaching practices, and internal factors, such as a child’s [developmental readiness] and interests” (Baroody, 1987a, p. 37). The importance of both external and internal factors is emphasized throughout the PSSM (NCTM, 2000). For instance, the following quotes are clear allusions to the latter factor: “Teachers of young students . . . need to be
knowledgeable about the many ways students learn mathematics” (p. 75). “Teachers must recognize that young students can think in sophisticated ways” (p. 77).

**Teachers Need to Understand the Whys and Hows of the Investigative Approach.** In regard to Point 1 above, early childhood and special education teachers need to understand the rationale for the reform movement (the standards-based investigative approach) and its recommended methods. This includes the psychological reasons why a purposeful, meaningful, and inquiry-based approach makes more sense than a traditional skills approach, or even the conceptual approach (as embodied, e.g., in the “California Standards,” California Department of Education, 1999; see, e.g., Baroody, with Coslick, 1998). Such knowledge will permit educators to substitute innovative methods for traditional ones in a thoughtful and reflective manner—that is, to use new educational tools flexibly, selectively, and adaptively. Parenthetically, it will also enable teachers to justify their methods effectively to interested others such as parents.

**Teachers Need to Be Able to Critically Analyze Activities.** In regard to Point 2, educators must be provided with more than a bag of tricks. They must be helped to construct the knowledge that enables them to distinguish between worthwhile activities and those that are not. This requires an extensive understanding of young children’s mathematical development. For instance, the still highly useful *Mathematics Their Way* program (Baratta-Lorton, 1976) includes a number of physical activities (e.g., hand-clapping, foot stomping, and finger snapping) to “provide experience saying one number with one motion” (p. 90). This one-to-one correspondence between a physical action such as finger pointing and uttering a count term is a key basis for object counting (enumeration).

Early childhood teachers who are knowledgeable about developmental research would know that children typically master this prerequisite for enumeration (Item VC3a in Table 1) quite early—about 2 years of age. Thus, they might choose the *Mathematics Their Way* activities described above as pre-counting (Transition 2) lessons for toddlers and children hampered by a developmental delay, because it would be an educative experience for them. They would not
choose to do this activity for most 3- to 6-year-olds, because it would be a mis-educative experience for these children—that is, it would be unnecessary for post-Transition 2 children (Baroody, with Coslick, 1998).

**Teachers Must Understand Children’s Knowledge and Thinking in Order to Provide Worthwhile Activities.** Point 3 above underscores the point made earlier in this chapter’s introduction. As the example in the previous section illustrates, the powerful and practical framework necessary to make effective teaching decisions must include knowledge of mathematical psychology (internal factors) as well as knowledge of content and methods (external factors). By understanding what young children know about these foundational concepts and what they can do with them, teachers can better incorporate developmentally appropriate activities to nurture their students’ mathematical development.

**What Mathematical Psychology Can Tell Early Childhood Educators**

A deep understanding of mathematical psychology can help educators decide what to teach, when to teach it, and how to teach. This point is illustrated by the example discussed in the next section.

**What to Teach.** As Parts I and II of this chapter suggest, young children display a surprising array of informal mathematical competencies. For example, not only do they seem ready to solve simple nonverbal and, later, verbal addition and subtraction problems, many preschoolers also seem capable of solving simple division (fair-sharing) problems. Not only do they seem capable of reasoning about whole numbers, but they may be capable of reasoning qualitatively about fractions.

**When to Teach.** As the Math Their Way example about one-to-one counting discussed in the previous section demonstrates, a familiarity with developmental psychology can be indispensable deciding when a particular concept or skill should be the focus of instruction.

**How to Teach.** Mathematical psychology can also provide educators invaluable clues about how to teach young children, in general, and how to help them learn specific skills, concepts, and inquiry competencies in particular.
A CASE IN POINT

Recent findings regarding children’s informal addition strategy of counting-on illustrates how psychological research can be used to justify the investigative approach in efforts to change beliefs about early childhood instruction and to illustrate a general (constructivist) teaching principle. That is, it can help teachers, administrators, and parents recognize how children can construct meaningful mathematical knowledge in a purposeful and inquiry-based manner, and why this is ultimately more beneficial to students than the traditional lecture-and-drill method. The recent psychological research regarding counting-on can also provide specific guidelines for facilitating key achievements in young children’s informal addition development by specifying what to teach, when to teach it, and, perhaps most importantly, how to teach it.

The General Principle: Guiding Children’s Mathematical Discoveries

Since psychological research has brought the relatively efficient counting-on strategy to the attention of educators, various efforts have been made to incorporate instruction on it into early childhood mathematics curricula and instruction. Often, these efforts involve direct instruction, such as modeling by a teacher or a textbook example and imitation by students (see, e.g., p. 92 of Eicholz et al., 1991).

I agree with Les Steffe (2000) that imposing such a strategy on children does not make sense. Children who do not understand the underlying conceptual basis for this strategy may learn counting-on by rote but not apply it when needed (i.e., they may forsake its use in favor of a more meaningful strategy). Such children may also misapply the meaningless strategy by, for instance, starting the keeping-track process too soon (e.g., for 5 + 3, counting: “5 [is one more], 6 [is two more], 7 [is three more]—the sum is 7”; Baroody et al., in press; Hopkins, 1998). Other children may simply reject the strategy outright and not use it.

Consider, for instance, the case of Felicia (Baroody, 1984), who typically used an abstract counting-all procedure (Achievement 5 in Table 3; e.g., for 3 + 5, counted: “1, 2, 3, 4, 5; 6 [is one more], seven [is two more], eight [is three more]—the sum is 8”). Although this child used an abstract counting-on or counting-on-like strategy (cf. Achievement 6 in Table 3) with large items,
such as $25 + 3$, when counting-on was modeled for her using single-digit combinations, she noted that you can’t add that way. Furthermore, she persisted in using counting-all strategies with such combinations, even after several demonstrations.

The case of Brianna illustrates the possible conceptual barriers to accepting and adopting a counting-on procedure (Baroody et al., in press). This kindergartner consistently considered a concrete counting-all strategy (Achievement 1 in Table 3) as “smart” and an abstract counting-on strategy as “not smart.” For instance, when the latter strategy was modeled using $6 + 8$, she explained, “You’re wrong. You started [counting the second addend] at nine. You are supposed to start at one.” This explanation is consistent with Fuson’s (1992) argument that to invent a verbal or abstract strategy, children must recognize that both addends can be represented in a single count (Achievement 4 in Table 3: “embedded integration for both addends”). For the other demonstration of counting-on with $7 + 5$, Brianna commented, “Maybe we should count out seven blocks [represent the first addend].” This explanation is consistent with Fuson’s (1992) observation that children must recognize that it is unnecessary to produce the first addend sequence (Achievement 6 in Table 3: “embedded cardinal-count principle”).

Nevertheless, I also agree with Karen Fuson’s (2000) commentary on Steffe’s (2000) paper—that it can be helpful and sometimes even necessary to promote the learning of counting-on. Fortunately, recent research suggests a way that teachers can guide children’s invention of this procedure, one that involves helping students make the conceptual breakthrough needed to understand the procedure. Put differently, the approach takes into account the concerns raised by both Steffe (2000) and Fuson (2000).

**Specific Guidance About What, When, and How**

Baroody (1995) found that children with normal or below-normal IQs typically began counting-on soon after discovering the number-after rule for $n + 1$ combinations: “The sum of a $n + 1$ combination is simply the number after $n$ in the counting sequence” (e.g., the sum of $5 + 1$ is the number after five: six). Bråten (1996), likewise, found this pattern with children with learning difficulties. Apparently, this induced rule provided a conceptual basis or scaffold for
counting-on. For example, children seem to reason that if the sum of $5+1$ is the number after \textit{five} in the counting sequence, then the sum of $5+3$ must be three numbers after \textit{five} in the counting sequence: “six, seven, eight.” Furthermore, as might be expected if the number-after rule served as a scaffold for inventing counting-on, the majority (6 of 10) of children described in the Baroody (1995) report first used this strategy (or first extended this strategy beyond $n+1$ or $1+n$ combinations) with $n+2$ or $2+n$ combinations, which require minimal attention to the keeping-track process.

What this research suggests is that a teacher can help children who have already achieved automatic number-after knowledge (when to teach) invent counting-on for themselves by prompting them to discover and discuss the number-after rule for $n+1$ combinations and providing purposeful opportunities to apply this discovery to computing $n+m$ combinations (what to teach; Baroody, with Coslick, 1998). In effect, the learning of this procedure can be achieved by indirect instruction that focuses on discovering patterns and relations and then applying them in somewhat novel contexts (how to teach). This approach is psychologically sound because it encourages children to construct their own understandings and procedures (Steffe, 2000). It is pedagogically sound because it involves children in mathematical inquiry and thinking (e.g., looking for patterns, using logical reasoning, communicating with peers). This approach also allows teachers to discharge their responsibility of promoting more advanced concepts and procedures in a nondidactic but a reasonably efficient manner (cf. Fuson & Secada, 1986; Secada, Fuson, & Hall, 1983).

Conclusions

In brief, as the discussion of how to teach counting-on illustrates, psychological research can help make a powerful and convincing case for the standards-based investigative approach by illustrating how it can be effective in promoting the learning of specific content, including basic skills (facts, procedures, and formulas), concepts or principles, and inquiry competencies (e.g., inductive and deductive reasoning; see Baroody, with Coslick, 1998, for additional examples). Psychological knowledge can also be invaluable to educators as a guide for how to implement
the investigative approach, in general, and to how to use it to teach specific content, in particular. It can also provide teachers the detailed knowledge about what, when, and how to guide the development of these specific mathematical competencies.

Even so, William James’ (1939) caution to educators about using psychological knowledge is still relevant:

I say moreover that you make a great, a very great mistake, if you think that psychology, being the science of the mind’s laws, is something from which you can deduce definite programmes and schemes and methods of instruction for immediate schoolroom use. Psychology is a science, and teaching is an art, and sciences never generate arts directly out of themselves. An intermediary inventive mind must make the application, by using its originality (pp. 7-8).

As suggested in Part I of this chapter, psychological knowledge is also constantly changing as more effective measures are devised, new facts are discovered, and theories change to accommodate the new evidence. As suggested in Part II, for instance, new evidence may reveal that 4-year-olds are simply responding to perceptual cues and really do not understand fraction addition and subtraction in a meaningful way.

Moreover, psychological findings are not the only basis for making educational decisions. Even if young children are capable of constructing a particular concept, such as fraction addition and subtraction, it does not necessarily follow that it should be taught. Educators must weigh the relative advantages and disadvantages of doing so because, among other practical considerations, teaching time is limited.

**Part IV: Using Standards as a Vehicle for Professional Development**

In this last part of the chapter, I make a case for using national and state early childhood standards as basis for helping educators, including teachers, supervisors, and curriculum developers construct the powerful developmental framework necessary to devise or implement effective child-centered mathematics instruction for all young children. I conclude with comments on the roles detailed psychological knowledge can and should play in early childhood mathematics
standards and the feasibility of including such information in these documents.

USING STANDARDS DOCUMENTS TO DISSEMINATE KNOWLEDGE ABOUT MATHEMATICAL LEARNING

The Rationale

National and state standards can and, I believe, should play a role in teacher professional development, the belief-changing and framework-building processes necessary to implement the investigative approach. This can be done, in part, by developing research-based standards that create new and more accurate expectations of young children. Including brief summaries of recent research or, better yet, vignettes illustrating their findings could also serve this purpose. A broad and concerted effort by the federal and state governments, the NCTM, and other interested parties should be undertaken to educate teachers, administrators, curriculum developers, and the public about the national and state early childhood standards and the evidence that supports them.

Using Numeral Reading and Writing As An Example

In an NCTM research pre-session talk (Baroody, 2000a), I used the development of numeral-reading and -writing skills to illustrate the value of psychological knowledge for early childhood and special education teachers. Below, I briefly summarize my key points, the reaction of one of the discussants of this paper, and some conclusions about the value of psychological knowledge to professional development.

Implications for Early Childhood Standards. Although young children typically construct a mental image and motor plan with little help, this process—as the discussion on pages 19 and 20 of Part II suggests—is not an uncomplicated process. To ensure that instructional efforts are well directed and children receive the guidance they need, teachers must be familiar with the details of this process. Furthermore, children with learning difficulties frequently need more than a little guidance but, unfortunately, often do not get it because their teachers are unaware of the psychological processes underlying numeral-reading and -writing skills.

Given the points just made, I (Baroody, 2000a) concluded that it is not sufficient for na-
tional or state standards for early childhood mathematics to simply state the goal, “Kinder-
gartners should master reading and writing numerals from 0 to 9.” Such a statement does not provide teachers any guidance on how to achieve the goal. I went on to conclude that national and state early childhood standards could and should be used a vehicle for helping teachers construct a better understanding of children’s mathematical learning (e.g., how they learn to read and write numerals).

A Counterargument. In her discussion of my (Baroody, 2000a) paper, Mary Lindquist noted that she never thought about sticks and loops when writing 6s and implied that the psychological model for learning how to read and write numerals was not important. She concluded that caring about and listening carefully to children is sufficient to overcome the problems we face in mathematics education. She further argued that (national or state) standards are not an appropriate forum to educate practitioners and not the place for detailed developmental information.

Psychological Knowledge: A Key But Sometimes Overlooked Component of Professional Development. Few would argue with the proposition that the professional development of teachers should include a solid grounding in mathematical content and pedagogy (Howe, 1999; Ma, 1999; NCTM, 1991). The third critical component of such development—an understanding of how children’s mathematical thinking and knowledge develop (e.g., Baroody, with Coslick, 1998; NCTM, 1991)—is not always considered in reform efforts (Baroody, 1987a; Kline, 1974; cf. Howe, 1999; Ma, 1999).

As I suggested earlier in this chapter, Lindquist was correct to imply that practitioners and those responsible for training them should not accept psychological theories or evidence uncritically. However, whether or not she has ever thought of a 6 as being composed of a stick and a loop is not relevant to gauging the value of the numeral-reading and -writing model I (Baroody, 2000a) discussed. For most adults and even many children, the mental images (knowledge of the parts and part-whole relations) and motor plans for numerals are nonconscious and may be largely or entirely non-verbal. “Sticks and loops” is merely an analogy for this underlying
knowledge—an analogy that can be useful to parents and teachers, especially when a child asks, “What does a 6 look like?” or “How do you make a 6?”

Even if a psychological model is not entirely accurate or complete—as is inevitably the case—its educational value depends on whether or not it yields useful predictions, insights, or guidelines. The numeral-reading and -writing model described earlier does effectively explain why, for instance, children are more prone to confuse some numerals (2 and 5 or 6 and 9) but not others, and why some children are prone to reversals, even with a model numeral in front of them. Moreover, this model is extremely helpful—as empirical evidence shows—in providing direction on how to overcome these difficulties. In fact, I have used the model with good results with typical children, including my own (Baroody, with Coslick, 1998), children diagnosed as learning disabled (Baroody & Kaufman, 1993), and those diagnosed as mentally retarded (Baroody, 1988a; Baroody, 1987a). This model and its supporting evidence is merely one example of the considerable body of psychological theory and research that has proven to be useful in teaching young children mathematics.

Caring about and listening carefully to children are unarguably crucial for effective teaching. However, by themselves, they are not enough; effective teaching also requires competence, which includes a powerful developmental framework (Baroody, with Coslick, 1998). Without such knowledge, teachers are not in a good position to help children no matter how much they care or how carefully they listen.

Consider the case of a second-grade teacher whose own son was diagnosed as learning disabled and was having considerable difficulty writing numerals. This woman, who cared deeply about her son, was unable to respond effectively to his pleas for help because her training did not include an effective model of numeral reading and writing. When he asked, for instance, “How do you make a 7?” she responded by drawing a 7. Unfortunately, such demonstrations were not enough for him to decipher where to start and in which direction to head, where to stop, and what to do next (i.e., to construct an accurate and complete motor plan). Despite his mother’s (and teachers’) best efforts, then, the boy continued to have numeral-writing difficul-
ties throughout the elementary grades and beyond. If his mother (and teachers) had had the theoretical framework to understand that his question (e.g., “How do you make a seven?”) was, in effect, a request for a motor plan, it is likely he could have been spared a great deal of unnecessary anguish.

If early childhood and special education teachers are ever to achieve the status of genuine professionals, they must be helped to secure an accurate and detailed understanding of young children’s mathematical learning. The proposed early childhood standards should be a beginning of this professional development. These documents are and should be inherently educational in nature. Why and how the standards should include specific psychological knowledge is discussed next.

THE ROLES OF DETAILED PSYCHOLOGICAL KNOWLEDGE IN EARLY CHILDHOOD STANDARDS AND THE FEASIBILITY OF ITS INCLUSION IN SUCH DOCUMENTS

Fuson (2000) raised several important questions about the inclusion of detailed psychological information in national or state standards documents: (a) What purpose would it serve? and (b) Is it practical? I address these questions in turn.

Purposes

Using detailed developmental knowledge to develop specific standards is useful, indeed, necessary for the following reasons:

• To serve as a guide for developing, maintaining, and evaluating the high-quality pre-service teacher education programs necessary to produce new but truly professional teachers who are capable of implementing an NCTM standards-based investigative approach.

• To serve as a guide for developing, maintaining, and evaluating the high-quality in-service teacher education programs necessary to upgrade or maintain a truly professional teaching corps that is capable of implementing an NCTM standards-based investigative approach.

• To develop and evaluate state, district, or school mathematics curricula that are consistent with an NCTM standards-based investigative approach.

• To develop and evaluate assessment means consistent with the NCTM (1991) Assessment
Standards at all levels (commercial, national, state, district, school, or classroom levels).

- To serve as a resource for curriculum coordinators or classroom teachers who are interested in implementing the NCTM standards-based investigative approach or who wish to further their professional development on their own.

Feasibility

Fuson (2000) suggested that developing specific standards would result in hundreds of statements and would make standards documents so overwhelming that they would, by and large, go unread. A practical solution is to have four levels of standards within each conceptual domain: (a) global standards that reflect the really big ideas; (b) general standards that indicate the big ideas; (c) specific standards that summarize the basic developmental components; and (d) detailed standards that delineate the developmental progression within each developmental component. An initial attempt to lay out these levels of standards the conceptual domain of number and operations is summarized in Figure 4.

Advocating a detailed level of standards should not be interpreted as suggesting a laundry list of lessons or units for teachers to implement. That is, I am not advocating that a teacher have one lesson or unit for each detailed standard. Typically, several detailed goals could be addressed in an integrated lesson or unit. For example, consider the game Animal Spots (Wynroth, 1986). On each turn, a player rolls a die with, for example, 0 to 5 dots on a side. The player counts the dots (goals N1.1.3b, N1.1.3c, N1.1.4d, and N1.1.4e in Table 3) or subitizes the number of dots (goal N1.1.4c), counts out a corresponding number of pegs (goals N1.1.5c and N1.1.5d) and puts the pegs (“spots”) into the holes of board cut out in the form of a leopard or giraffe. The first player to fill all the holes in his or her animal board is the winner.

Note also that playing this game is consistent with an investigative approach. Children are engaged in an activity that is purposeful to them (i.e., it involves learning and practicing mathematical competencies in an inherently real, interesting, and meaningful manner). Because it involves using mathematical competencies in context, children can better understand why and how the competencies are used (i.e., the games provide a basis for meaningful or conceptual
learning). For example, if one child counts five dots and another sees four, the discrepancy can provide an opportunity for discussing one-to-one counting and keeping-track strategies (strategies for distinguishing between already-counted items and items yet to be counted). The discrepancy can also provide an opportunity for engaging in mathematical inquiry. For instance, a teacher might ask if both answers could be correct, how the group could decide which was correct, and why the incorrect answer is incorrect and how it came about.

A main standards document could lay out the first three levels of standards. That is, it could consist of a relatively few global standards, a number of general standards, and include tables, figures, or appendices that summarize the specific standards. This document could also indicate where an interested party could go for the more-detailed standards. The detailed standards could be laid out in a series of supplemental standards (e.g., one for each of the global standards). In addition to or in place of printed supplemental standards, a web site could include the specific standards for each specific standard. Indeed, the web site could be designed to start with the global standards and provide increasing specific standards. Visitors would explore an area as deeply as they needed or desired.

Conclusion

In conclusion, federal and state governments now spend millions of dollars on special education to, for example, provide children with special needs small classes and, in many cases, a personal aide. Unfortunately, the teachers of these smaller classes and the personal teaching aides—by and large—do not have a deep understanding of how special children learn mathematics. The result is that most mathematics instruction of special children is ineffective. Put differently, much of the federal and states’ investment in special education is simply wasted.

We risk the same result with our efforts to improve early childhood instruction, if a concerted effort is not made to help early childhood educators construct a powerful and practical framework that includes a deep understanding of how young children learn mathematics. In order to help remedy the insufficient attention paid to the mathematics instruction and learning of young children, particularly those with special needs, the National Council of Teachers, state
education departments, and the Federal government—through their publications, the proposed national and state early childhood mathematics standards), and other efforts—need to help early childhood and special pre- and in-service teachers construct an accurate and extensive knowledge of the mathematical teaching and learning of all children.

Helping pre- and in-service teachers construct a deep understanding of the mathematics they need to teach, how children learn mathematics, and how to foster this learning effectively is necessary for elevating teaching to the level of a true profession (one comparable to medicine or law) and for the success of the current reform movement. Many anti-NCTM-standards proponents, including some supporters of the California Standards, believe that it is not practical or even possible to help teachers achieve professional-level knowledge—a deep understanding of mathematics, psychology, and pedagogy. For this reason, they propose a different approach to reform: skills-focused standards, “teacher-proof” curricula, high-stakes standards-based testing, and accountability. The aim is to eliminate incompetent teachers or ineffective schools (rather than support their re-development).

I was disturbed to hear again and again at our October meeting in Dallas that the EC standards should not be too complicated because, for example, teachers would be overwhelmed. By advocating “dumbed down” standards, we are conceding that anti-NCTM-standards proponents’ (and Brownell’s [1935]) premise is correct, namely that the vast majority of teachers are not bright or sufficiently motivated enough to acquire the knowledge necessary to implement sophisticated instruction, such as the investigative approach, effectively. (I am ashamed that there were times during the meeting when I concluded that this pessimistic conclusion was, in fact, a realistic assessment of the situation.) By focusing on global and general standards to the exclusion of specific and detailed standards, we help to set in concrete what is, not on what could or should be.

Developing a simplified list of goals for each grade level has its uses and should be done. However, it should be clearly tied to big mathematical ideas and to more specific and detailed research-based goals. The latter is necessary to counter the charges of anti-NCTM-standards
proponents that current reform efforts are merely “fuzzy math” and based on ideology rather than science. In brief, the development of a coherent multi-level set of standards as illustrated in this paper is necessary to empower teachers to foster the mathematical power of all children.
Footnotes

1 Starkey’s (1992) ingenious device for testing nonverbal addition and subtraction knowledge controlled for one type of touch cues but not another (J. Huttenlocher, personal communication, May 26, 2000). His task involved putting a certain number of balls into a tube one at a time and then changing the initial number by adding or taking away balls one at a time. A participant saw the initial action(s) and subsequent change action(s) but could not see the outcome. The dependent measure was the number of times a child reached into the tube, which served as a nonverbal indication of the participant’s expectation of the final outcome.

The tube had a false bottom so that when a child reached into the tube, only one ball was present. This controlled for gauging by touch whether there were other balls in the tube that required additional reaches. Huttenlocher et al. (1994) pointed out that feeling only one ball, however, provided a touch cue that favored items with an answer of one, because participants might conclude there was only a single ball in the tube and that further reaches were unnecessary. They noted, for instance, that Starkey’s (1992) participants were successful an astounding 64% of the time on the 4 – 3 item but were successful only 14% of the time on the 4 – 1 item, which in several ways is the easier of the two items. (Taking away one presumably is easier than taking away three because there are fewer changes overall of which to keep track. This simpler process may put less demand on a child’s attentional resources and working memory.

2 Wynn’s (1990, 1992) operational definition of understanding that a number word refers to a specific numerosity is based on a production task (counting out a specified number of items from a larger collection of items. This task requires a relatively advanced cardinal concept (what Fuson, 1988, calls a cardinal-count concept; Baroody, 1987a), as well as processes involving working-memory (remembering the target and matching a count to the target; e.g., Resnick & Ford, 1981). As a result, a production task may underestimate young children’s understanding of number as a referent to numerosity. Although further research is clearly needed, Wynn’s (1990, 1992) conjecture that a construction of a verbal-based number concept does not
occur quickly and may not be guided directly or perhaps, even indirectly, by children’s nonverbal number knowledge may still be correct.

3 Constance Kamii, in her discussion of my paper presented at the Conference on Standards for Preschool and Kindergarten Mathematics Education (Baroody, 2000b), noted that Piaget distinguished between the form of representation and mental structures (e.g., the conceptual content represented) and that he argued that the latter was more important. She concluded her comments by dismissing the mental model as unimportant and useless. Although knowing only the form of number representation, for instance, does not necessarily indicate the extent of a child’s understanding of number, it is, in fact, important and useful to understand. For one thing, the type of representation may limit a child’s understanding of number or their ability to reason about number in critical ways. As will be illustrated later in the text, the development of more advanced number representation can provide children with a more powerful means of thinking about and using numbers.

Indeed, Piaget (1951) himself devoted considerable effort to describing the development of a symbolic function and how this development built on but differed from sensory-motor intelligence. As Ginsburg and Opper (1969) noted:

“The ability to form mental symbols is an achievement of great magnitude. In the sensorimotor period this capacity was lacking . . . . By contrast, the older child can use mental symbols to stand for absent events or things. Things no longer need to be present for the child to act on them. In this sense, the ability to symbolize eventually liberates the child from the immediate present” (p. 78).

For a discussion of how representational intelligence (through the use of the symbolic function) differs from sensory-motor intelligence, see Flavell (1963, pages 151 and 152, including the footnote at the bottom of page 151).

4 Unlike the mental model, though, Piaget (1965) believed that children did not construct an operational or genuine understanding of one-to-one correspondence until they were about seven. This was signaled by the ability to conserve number—an achievement that occurs well after
children have learned verbal- and object-counting skills. In Piaget’s (1965) view, these counting skills were merely learned and used by rote, a view subsequent research largely suggests is inaccurate.

In her critique of my paper presented at the Conference on Early Math Standards (Baroody, 2000b), Connie Kamii noted that reversing numerals was common among pre-second-graders and not a cause of concern because children naturally outgrow such errors. She concluded that students do not need help writing numerals correctly. In general, Kamii’s comments are all true. However, many children recognize for themselves that they are writing numerals incorrectly (e.g., in reverse), are puzzled or concerned about this, and ask for help (e.g., How do you make an 8?). Teachers should recognize that such questions are a request for an accurate motor plan. Furthermore, children with learning difficulties frequently do not spontaneously discover or invent correct motor plans and, in such cases, reversals (or other errors) may persist for years (Baroody, 1987a, 1988a; Baroody & Kaufman, 1993). For such children, remedial efforts, including help with constructing a motor plan, are necessary.

Coincidentally, a request from Diane Goodman, a Senior Editor at Riverside Publishing (via Doug Clements and Danny Breidenbach of NCTM), several weeks after the NCTM presession underscores the value of the psychological model for numeral-reading and –writing discussed above and in the text. The question asked was, “Which format of the 4 [open versus closed] is most (sic) appropriate for grades K-2 and why?

Without a theoretical model, the choice between an open 4 and a closed 4 is arbitrary or difficult to make. It follows from the model discussed that the open 4 is somewhat more preferable, because it does not involve a diagonal. Although this may not be a factor in forming a mental image of the numeral or reading it, the mental image children do form may impact the ease with which they master a motor plan. Writing numerals (and letters) composed only of horizontal and vertical lines is easier than doing so for those that involve a diagonal. A motor plan for the latter requires orienting in two directions at once. For instance, for the closed 4, the plan for the diagonal would be to start at the upper right and slant down to the lower left
(would involve both up-down and right-left directions). Theoretically, the most efficient way to teach reading and writing 4s, then, would be to use open 4s. This way, children’s mental image would be consistent with the easier motor plan for written 4s.

6 Note that, even before children construct a concrete counting-all procedure, they may use exact verbal subitizing to represent addends and to determine a solution (Level 1C in Figure 2 discussed earlier) and then use this noncounting process to accomplish one of these aims and a counting process to do the other (e.g., Level 2a in Figure 2). In some cases, then, the shortcut described in Achievement 2 in Table 3 may actually come before Achievement 1. Pilot work found one such example.

7 An almost identical sentence appeared in the *PSSM* draft (NCTM, 1998), except that *recall* was used instead of *know*, and *facts* was used instead of *combinations*. Although these changes appear to contradict the argument that knowing is not being contrasted with fluency, they seem to be a Clintonian gambit to use words ambiguously when compared with the clear-cut statement from page 78 in Chapter 4 quoted earlier.

8 Three problems with Irwin’s (1996) methodology, however, render her results inconclusive. It is not clear that she really measured part-whole understanding at Resnick’s (1992) protoquantities level). For example, 4-year-olds compared an experimenter’s collection of three items with their own three-item collection, which was subdivided into parts of two items and one item. These collections could easily have been non-verbally subitized, verbally subitized, or subvocally or surreptitiously counted. In other words, children could easily have been using any of several exact-number processes (Level 1B, Level 2, or higher levels in Table 1), instead of an inexact process that characterizes Resnick’s (1992) protoquantities level (Level 1 in Table 1). Second, it is not clear whether she was actually measuring part-whole-knowledge or something else with the so-called protoquantities task. Her participants could have been responding correctly to the co-variation trials simply by noticing that an item was added or taken away rather than considering the effects on the child’s whole relative to the tester’s (cf. Brush, 1978). In effect, the children might have been interpreting the task in terms of a change add-to view of
addition, not in terms of part-whole view. Third, the putative variable of interests, namely the conceptual level, was not the only way the protoquantities-level task differed from the quantities- and numbers-level tasks. With the first task, a child could readily see (by nonverbal or verbal subitizing or by subvocal counting) that a collection of items was split evenly between the tester and him- or herself. With the quantities-level task, for example, the child was asked to give the tester a specified number of items, which the latter then hid in one hand. The tester next placed an unspecified number of items in the other hand, which was then operated on in various ways. The participant could, then, not be sure that the collections in the two hands, were, in fact, equal and may not have been thinking in terms of a specific number of items. Furthermore, unlike the so-called protoquantities task, children did not see both parts in the quantities task.

A popular view is that “play is children’s work” (NCTM, 2000, p. 74; see also Bruner, Jolly, & Sylva, 1976). Dewey’s (1963) distinction between educative and mis-educative experience can be interpreted to mean that not all play is the former—of equal value developmentally. Although this qualification is important for practitioners to keep in mind, common sense dictates that play for the sake of fun is valuable for young children and has a place in early childhood education. In other words, some balance between play for learning and play simply for joy seems a reasonable goal for young children.

Without a powerful psychological framework to guide them, teachers may blindly follow teacher guide or curricula. In Mathematics Their Way, for instance, Baratta-Lorton (1976) suggested having children copy a numeral with finger motions in the air or in the palm of their hand. Unfortunately, such a procedure leaves no visible record for a child or a teacher to evaluate the child’s successful execution of a motor plan. As a result, a child who is using an inaccurate or incomplete motor plan may not receive the feedback needed to prompt a correction to this plan. Furthermore, the authors of one elementary mathematics education textbook (D’Augustine & Smith, 1992) recommended delaying numeral-writing instruction until first grade—until after children developed the fine-motor coordination necessary for this skill. This
advice disregards the fact that the numeral-writing difficulties of kindergartners are typically due to not having learned a complete and accurate motor plan, not the lack of fine-motor coordination.
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Appendix A:

Some Resources on Early Childhood Mathematics Education


**Figure 1: Proposed Nonverbal Representations of Number**

<table>
<thead>
<tr>
<th>Representation</th>
<th>Possible Underlying Mechanism</th>
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<tbody>
<tr>
<td>PERCEPTUAL CUES</td>
<td>1. Use of length, area, or density to numerosity</td>
</tr>
<tr>
<td>(Use of qualitative or perceptual cues that often co-vary with numerosity)</td>
<td>(2) Use of contour length (Clearfield (\delta))</td>
</tr>
<tr>
<td>ITEM SUBITIZING</td>
<td>Pattern recognition: (\bullet) (a point) = 1; (\times) (a grouping) = 3 (Mandler &amp; Shebo, 1982).</td>
</tr>
<tr>
<td>(Accurate and fast recognition of small numerosities—collections of 1 to 3 or 4)</td>
<td>Recognition of recurring spatial or temporal information (Glasersfeld, 1982).</td>
</tr>
<tr>
<td>ITEM + EVENT SUBITIZING</td>
<td>(1) Finger of instantiation (FINST) assigned for each key feature—the by-product spatial individuation process (Kahneman 1992; Trick &amp; Pylyshyn, 1994). These markers represent numerosity and, as full-functional mental tokens, can be manipulated mentally (Simon, 1997; Fenner, &amp; Flatt, 1999).</td>
</tr>
<tr>
<td>(Accurate and fast number recognition of small simultaneous objects or sequential objects)</td>
<td>(2) Imaginistic mental model is constructed by the symbolic representation of individual objects—e.g., (\bullet\bullet\bullet) or (\bullet\bullet\bullet) might be represented as (\bullet\bullet\bullet).</td>
</tr>
<tr>
<td>OBJECT TOKEN</td>
<td>Unlike the models above, which do not use counting mechanisms, numerical information is represented nonverbally using a counting algorithm. For example, the number 5 can be represented by assigning the tags @ fashion and then using the last tag # to represent the total (Gallistel, 1990; Gallistel &amp; Gelman, 1992).</td>
</tr>
<tr>
<td>(Symbolic/mental/imaginistic representation of individual objects)</td>
<td></td>
</tr>
</tbody>
</table>
Table 1: A Comparison of Two Models of Mathematical Development

<table>
<thead>
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<tbody>
<tr>
<td>(kinds of conceptual entities)</td>
<td></td>
</tr>
<tr>
<td>[Level 0: protoquantitative thinking involving qualitative reasoning about non-quantified entities]?</td>
<td></td>
</tr>
<tr>
<td>Level 1: protoquantitative thinking involving qualitative reasoning about inexact (uncounted but perceptually estimated) quantities</td>
<td></td>
</tr>
<tr>
<td>✡ Transition from inexact nonverbal representation to an exact nonverbal representation not discussed ✡</td>
<td></td>
</tr>
<tr>
<td>[Level 1A: first transitional subphase to quantitative thinking involving qualitative reasoning about exact, nonverbally represented quantities]?</td>
<td></td>
</tr>
<tr>
<td>[Level 1B: second transitional subphase to quantitative thinking involving quantitative reasoning about exact, nonverbally represented quantities]?</td>
<td></td>
</tr>
<tr>
<td>Level 1C: third transitional subphase to Level 2 thinking involving quantitative reasoning about subitized (exact, nonverbally represented) quantities and their number labels]?</td>
<td></td>
</tr>
<tr>
<td>Level 2: (counting-based) quantitative thinking</td>
<td></td>
</tr>
<tr>
<td>Level 3: numerical reasoning?</td>
<td></td>
</tr>
<tr>
<td>Level 4: abstract reasoning (generalizing)?</td>
<td></td>
</tr>
<tr>
<td>[Transition from verbal to written representation not discussed]</td>
<td></td>
</tr>
<tr>
<td>Level 3: numerical reasoning?</td>
<td></td>
</tr>
<tr>
<td>Level 4: abstract reasoning (generalizing)?</td>
<td></td>
</tr>
<tr>
<td>Note. Resnick (1992) did not propose sublevels for Level 1 (protoquantitative thinking); Levels 0, 1A, 1B, and 1C (shown above in brackets). These four levels undoubtedly extend beyond Transition 1, 2, or even 3. Level 1C takes into account the possibilities above in brackets. Note also that the relations among Resnick’s four levels may not be entirely linear. For example, the development domain may prompt lower-level thinking in this domain and other domains.</td>
<td></td>
</tr>
</tbody>
</table>

Figure 2: Hypothesized Relations Among the Levels Listed in Table 1, Number Competencies, and Addition/Subtraction Skills

<table>
<thead>
<tr>
<th>Level</th>
<th>Type of number representation</th>
<th>Type of subitizing (using 3 items as an example)</th>
<th>Type of production (using 3 items as an example)</th>
<th>Type of reasoning</th>
<th>Adding (using 1 item + 2 more as an example)</th>
<th>Taking away (using 1 item from 3 as an example)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>inexact nonverbal (perceptual)</td>
<td>inexact nonverbal production</td>
<td>Qualitative (???, ???, ? or</td>
<td>estimated sum (???, ???, ? or</td>
<td>estimated difference (???, ???, ? or</td>
<td></td>
</tr>
</tbody>
</table>

Part Two, Section 2. Do not share or cite. 2-2-3 p. 92 4/18/01
cue such as total perimeter

<table>
<thead>
<tr>
<th>Level</th>
<th>Type</th>
<th>Nonverbal Matching</th>
<th>Quantitative Nonverbal Production</th>
<th>Qualitative Nonverbal Production</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A</td>
<td>exact</td>
<td>nonverbal (???)</td>
<td>estimated sum (???, or ???)</td>
<td>estimated difference (?)</td>
</tr>
<tr>
<td>1B</td>
<td>exact</td>
<td>nonverbal (???)</td>
<td>nonverbal exact sum (?)</td>
<td>nonverbal exact difference (?)</td>
</tr>
<tr>
<td>1C</td>
<td>exact</td>
<td>verbal (??? ✿ &quot;3&quot;)</td>
<td>exact quasi-verbal addition (??? + &quot;3&quot;)</td>
<td>exact quasi-verbal subtraction (?? + &quot;2&quot;)</td>
</tr>
<tr>
<td>2</td>
<td>exact</td>
<td>verbal (??? ✿ &quot;3&quot;)</td>
<td>a. subitizing-based, exact verbal production (Request for &quot;three&quot; ✿ puts out ??? simultaneously)</td>
<td>a. counting-based solutions using subitized quantities to represent addends.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>b. counting-based exact verbal production (Request for &quot;three&quot; ✿ counts out ??? sequentially)</td>
<td>b. counting-based solutions using counted-out collections to represent addends.</td>
</tr>
</tbody>
</table>

*Note that inexact matching (creating a collection that approximates a visible model collection) may precede inexact nonverbal production (creating an approximation of a hidden model collection that once was visible). Furthermore, exact nonverbal matching may follow inexact nonverbal production but appears before Transition 1 and exact nonverbal production (creating an exact copy of a hidden collection that was once visible; Level 1A above).

Table 2: A Summary of Preschool Number and Arithmetic Development

<table>
<thead>
<tr>
<th>Item</th>
<th>Appx Age</th>
<th>Nonverbal Number Concept/Skill</th>
</tr>
</thead>
<tbody>
<tr>
<td>NN1</td>
<td>Transition 1: Becomes capable of nonverbally representing small collections picture, mental marker, or some other means (Huttenlocher, Jordan, &amp; Levine, 1992)</td>
<td>Nonverbal object-counting competencies develop and underlies the following skills.</td>
</tr>
<tr>
<td>NN1a</td>
<td>2 - $\frac{21}{2}$</td>
<td>• Nonverbally subitizes and produces one or two items (e.g., after seeing a collection which is then hidden from view, creates a matching collection).</td>
</tr>
</tbody>
</table>

Part Two, Section 2. Do not share or cite. 2-2-3  p. 93  4/18/01
NN1b 3 • Nonverbally identifies as equivalent or nonequivalent static (simultaneously geneous collections, both consisting of a few (1 to about 4) identical objects Siegel, 1973). For instance, can identify ••• and ••• as equal and different fr

NN1c 3\frac{1}{4} - 4 • Nonverbally subitizes and produces up to 4 items.

NN1d 3 - 3\frac{1}{2} • Nonverbally matches equivalent sets of disks and dots—two homogeneous : highly similar items (Mix, 1999b).

NN2 Transition 2: the assimilation of verbal-based number representations to no: the capacity for making equivalent judgments in the following ways:

NN2a 3\frac{1}{2} - 4 2. Nonverbally matches equivalent sets of shells and dots—two homogeneous similar items (Mix, 1999b). (Requires minimal object-counting competence. appear to require accurate enumeration skill with small collections.)

NN2b 3\frac{1}{2} - 4 4. Nonverbally identifies a static collection (e.g., 3 dots) with a sequentially pr 3 successively presented dots; Mix, 1999a; Mix, Huttenlocher, & Levine, 199

NN2c 4 6. Nonverbally makes a cross-modal (auditory-visual) match (e.g., matches 3 Mix et al., 1996).

NN2d 4 8. Nonverbally matches a static collection such as three dots to a sequentially-1 three light flashes or three jumps by a puppet (Mix, 1999a).

NN2e 4 - 4\frac{1}{2} 10. Nonverbally matches equivalent sets of random objects and dots—a heterog a set of dissimilar items (Mix, 1999b).
Table 2 continued

<table>
<thead>
<tr>
<th>Item</th>
<th>Appx Age</th>
<th>Verbal-Based Counting Concept/Skill</th>
</tr>
</thead>
<tbody>
<tr>
<td>VC1</td>
<td></td>
<td>Chanter: Unbreakable, undifferentiated chain—i.e., “sing-song” or verbal chant of in words (e.g., onetwothree). Begins to use number words in sequence:</td>
</tr>
<tr>
<td>VC1a</td>
<td>1$^{1/2}$ - 2$^{1/2}$</td>
<td>• Strings together numbers to form an (unconventional) counting sequence such as “one, two, one, two”</td>
</tr>
<tr>
<td>VC1b</td>
<td>1$^{1/2}$ - 2$^{1/2}$</td>
<td>• Includes &quot;one&quot; in the counting string.</td>
</tr>
<tr>
<td>VC2</td>
<td></td>
<td>Rote counter: Unbreakable, differentiated chain—i.e., distinguishes among number words in sequence but cannot count-on (e.g., may count, &quot;one two, three, four, but cannot start than &quot;one&quot;).</td>
</tr>
<tr>
<td>VC2a</td>
<td>2 - 3$^+$</td>
<td>• Cites distinct numbers in sequence but merely imitates object counting by with collection and chanting a sequence of numbers.</td>
</tr>
<tr>
<td>VC2b</td>
<td>2 - 4</td>
<td>• Cites numbers in the correct sequence to &quot;ten.&quot;</td>
</tr>
<tr>
<td>VC3</td>
<td></td>
<td>Enumerator of small collections (1 to about 4 or 5): Recognizes that counting sequence must be repeated to represent a collection.</td>
</tr>
<tr>
<td>VC3a</td>
<td>2</td>
<td>• Recognizes and labels with a number collection of one or two. This verbal-sketch serves as the transition between nonverbal subitizing and enumeration (e.g., 1982)</td>
</tr>
<tr>
<td>VC3b</td>
<td>2 - 3</td>
<td>• Pairs a number word of the count sequence with each item of a small collection by repeating the count (Wagner &amp; Walters, 1982). For example, a collection of four items and respond to how-many question by re-reciting “four.”</td>
</tr>
<tr>
<td>VC3c</td>
<td>2$^{1/2}$ - 3$^{1/2}$</td>
<td>• Verbally subitizes collections of up to about four or five items.</td>
</tr>
<tr>
<td>VC3d</td>
<td>2$^{1/2}$ - 3$^+$</td>
<td>• Constructs the count-cardinal concept (i.e., recognizes that the last number items in a collection also represents the total). Child now responds to the how-many question with the cardinal designation of a collection (e.g., “There are four”).</td>
</tr>
<tr>
<td>VC3e</td>
<td>3$^{1/2}$ - 4$^{1/2}$</td>
<td>• Constructs a number-identity or constancy principle (i.e., recognizes that collection applies to any collection of a particular size by constructing.</td>
</tr>
<tr>
<td>VC3f</td>
<td>4 - 5</td>
<td>• Constructs an order-irrelevance principle (i.e., explicitly recognizes that the order-ordered counts of a collection will result in the same cardinal value; e.g., 1993).</td>
</tr>
</tbody>
</table>

VC4

Verbal producer of small collections (1 to about 4 or 5): Recognizes that counting sequence must be repeated to represent a collection.
<table>
<thead>
<tr>
<th>Item</th>
<th>Appx Age</th>
<th>Verbal-Based Counting Concept/Skill</th>
</tr>
</thead>
<tbody>
<tr>
<td>VC4a</td>
<td>2 - 3</td>
<td>• In response to a verbal request for &quot;one&quot; or &quot;two&quot; items, uses subitizing to create a finger pattern to represent the number nonverbally (e.g., in response to a request for &quot;one&quot;, immediately and simultaneously—without counting—takes two blocks; see, e.g., Wilkins &amp; Baroody, 2000). This quasi-verbal production facilitates the transition between nonverbal production and verbal production (Baroody, 2000).</td>
</tr>
<tr>
<td>VC4b</td>
<td>2½ - 3½</td>
<td>• Quasi-verbal production of up to 4 or 5 items via subitizing or finger patterns</td>
</tr>
<tr>
<td>VC4c</td>
<td>2½ - 3½</td>
<td>• Constructs the cardinal-count concept (i.e., recognizes that a cardinal label as &quot;four&quot; is equivalent to actually counting the collection).</td>
</tr>
<tr>
<td>VC4d</td>
<td>2½ - 3½</td>
<td>• This concept and the ability to match a count to a target value stored in working memory provide the basis for verbal production of sets, such as responding to a request to count blocks from a pile of blocks (Baroody, 1987a).</td>
</tr>
<tr>
<td>VC5</td>
<td>Flexible verbal counter: Use of the number sequence extended in various ways.</td>
<td></td>
</tr>
<tr>
<td>VC5a</td>
<td>4 - 5</td>
<td>• At the breakable-chain level of counting, begins to count from any number in the sequence, permits counting-on from a number (e.g., counting &quot;four, five, six, seven . . .&quot; after another (without counting from &quot;one&quot;).</td>
</tr>
<tr>
<td>VC5b</td>
<td>4½ - 6</td>
<td>• At the bidirectional chain level of counting, counts in the opposite direction the collection learned—i.e., backwards.</td>
</tr>
<tr>
<td>VC5c</td>
<td>5 - 7</td>
<td>• Skip counts (e.g., counts by twos or fives).</td>
</tr>
</tbody>
</table>
Table 2 continued

<table>
<thead>
<tr>
<th>Item</th>
<th>Approx Age</th>
<th>Appx Age</th>
<th>Item-Based Number Sense</th>
<th>Verbal-Based Number Sense</th>
</tr>
</thead>
</table>

Note that comparisons involving "less" trail behind those involving "more."
<table>
<thead>
<tr>
<th>Item</th>
<th>Appx Age</th>
<th>Written Representation of Number (Form)</th>
</tr>
</thead>
<tbody>
<tr>
<td>WN1</td>
<td>4 - 6</td>
<td>Item</td>
</tr>
<tr>
<td>WN2</td>
<td>4 1/2 - 7</td>
<td>Constructs a motor plan of the numerals 0 to 10, which consists of a step-by-step plan of execution for translating a mental image into motor actions. This allows children to:</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Constructs a mental image of the numerals 0 to 10, which consists of knowledge about parts and part-whole relations. This permits children to:</td>
</tr>
<tr>
<td>WN3</td>
<td>6</td>
<td>Identifies the larger of two written numerals.</td>
</tr>
<tr>
<td>WN4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>WN5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>WN6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>WN7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>WN8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>WN9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>WN10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2 continued
Table 2 continued

<table>
<thead>
<tr>
<th>Item</th>
<th>Appx Age</th>
<th>Written Representation of Number (Function)</th>
</tr>
</thead>
</table>

Transition 3: Collections and verbal numbers can be represented by written symbols.

(Revised work by Hughes, 1986, suggested the following developmental progression in form:

- Functional symbols: Written symbols used to represent cardinal value of a collection, become genuine cognitive tool

- Nonfunctional symbols:
  - Written symbols (iconic or conventional) are produced simply to correspond with a spoken number
  - Idiosyncratic symbols (understandable only to creator)

Previous work by Munn (1998) suggested the following developmental progression in function:

- Written symbols not used
- Written symbols (iconic or conventional) are produced simply to correspond with a spoken number
- Written symbol used to represent cardinal value of a collection; becomes a genuine cognitive tool

-Nonfunctional symbols:
  - Idiosyncratic symbols (understandable only to creator)
  - Pictographic symbols (drawings of objects) or iconic symbols (e.g., drawing three circles to represent three items)
  - Conventional symbol use (e.g., writing the numeral 3 to represent three items)
<table>
<thead>
<tr>
<th>Item</th>
<th>Appx Age</th>
<th>Nonverbal Arithmetic: Addition &amp; Subtraction Problems Involving 1, 2, 3, 4 (e.g., Huttenlocher et al., 1994), I or Post-transition 1 (i.e., they imagine adding one item to another or taking away one from two; e.g., Huttenlocher et al., 1994).</th>
</tr>
</thead>
<tbody>
<tr>
<td>NA1</td>
<td>3</td>
<td>Solves nonverbal addition and subtraction problems involving 1 or nonverbal subtraction problem involving 2.</td>
</tr>
<tr>
<td>NA2</td>
<td>4</td>
<td>Solves nonverbal addition and subtraction problems involving 1, 2, 3, or subtraction involving 1 or nonverbal subtraction problem involving 2.</td>
</tr>
<tr>
<td>NA3</td>
<td>5</td>
<td>Solves nonverbal addition and subtraction problems involving 1 or nonverbal subtraction problem involving 2.</td>
</tr>
<tr>
<td>NA4</td>
<td>6</td>
<td>Solves nonverbal addition and subtraction problems involving 1, 2, 3, or subtraction involving 1 or nonverbal subtraction problem involving 2.</td>
</tr>
</tbody>
</table>

Table 2 continued
### Table 2 continued

<table>
<thead>
<tr>
<th>Item</th>
<th>Appx Age</th>
<th>Concept/Skill</th>
</tr>
</thead>
<tbody>
<tr>
<td>VA1</td>
<td>5-8</td>
<td>Connects verbally presentation of addition and subtraction to their conceptual understanding</td>
</tr>
</tbody>
</table>
Table 2 continued

<table>
<thead>
<tr>
<th>Item</th>
<th>Appx Age</th>
<th>VA6</th>
<th>VA7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discovers other relations that allow the invention of &quot;thinking strategies&quot; or heuristics (e.g., the doubles-plus-one heuristic) as a 7 + 8 → 7 + [7 + 1] → [7 + 7] + 1 = 14 + 1 = 15). See Box 5.6 on pages 5-31 to 5-33 of Baroody, with Coslick, (1998) for a detailed discussion of arithmetic thinking strategies. Note that learning thinking strategies for subtraction is considerably more difficult than that for addition (see, e.g., Baroody, 1999; Fuson, 1992; Siegler, 1984). Masters basic addition and subtraction combinations by internalizing relational as well as factual knowledge.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Item</th>
<th>Appx Age</th>
<th>VA6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Masters basic addition and subtraction combinations by internalizing relational as well as factual knowledge.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Item</th>
<th>Appx Age</th>
<th>VA6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discovers other relations that allow the invention of &quot;thinking strategies&quot; or heuristics (e.g., the doubles-plus-one heuristic) as a 7 + 8 → 7 + [7 + 1] → [7 + 7] + 1 = 14 + 1 = 15). See Box 5.6 on pages 5-31 to 5-33 of Baroody, with Coslick, (1998) for a detailed discussion of arithmetic thinking strategies. Note that learning thinking strategies for subtraction is considerably more difficult than that for addition (see, e.g., Baroody, 1999; Fuson, 1992; Siegler, 1984). Masters basic addition and subtraction combinations by internalizing relational as well as factual knowledge.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Item</th>
<th>Appx Age</th>
<th>VA6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Masters basic addition and subtraction combinations by internalizing relational as well as factual knowledge.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table 2 continued

<table>
<thead>
<tr>
<th>Item</th>
<th>Appx Age</th>
<th>Part-Part-Whole</th>
</tr>
</thead>
<tbody>
<tr>
<td>PW1</td>
<td>$\frac{21}{2}$</td>
<td>Nonverbally identifies both a whole and its parts (Boisvert et al., 1999).</td>
</tr>
<tr>
<td>PW2</td>
<td>4</td>
<td>Recognizes the effects of transformations to a part or parts on an uncounted whole (Boisvert et al., 1999).</td>
</tr>
<tr>
<td>PW2a</td>
<td></td>
<td>Recognizes that incrementing a part increases its whole; decrementing it decreases that part’s contribution to the whole (variation principle).</td>
</tr>
<tr>
<td>PW2b</td>
<td></td>
<td>Recognizes that taking away an item from one part and adding it to the other part does not increase/decreases the whole (compensation principle supported by a move schema).</td>
</tr>
<tr>
<td>PW2c</td>
<td></td>
<td>Recognizes that taking away an item from one part and adding a different item to another part does not change the whole (compensation principle supported by a balance schema).</td>
</tr>
<tr>
<td>PW3</td>
<td>4 - 6</td>
<td>Recognizes the effects of transformations to a part or parts on a counted whole (Boisvert et al., 1999). Two-thirds of 4-year-olds, two-thirds of 5- and 6-year-olds, and (nearly) all 6-year-olds recognize the effects of transformations to a part or parts on a counted whole.</td>
</tr>
<tr>
<td>PW4</td>
<td></td>
<td>Nonverbal (protoquantities-level) understanding of class inclusion.</td>
</tr>
<tr>
<td>PW4a</td>
<td>?</td>
<td>Succeeds on verbal (qualitative-reasoning) class-inclusion tasks that emphasize understanding of class inclusion (Fuson et al., 1998; Markman, 1973) or otherwise control for extraneous difficulty (McCorgray, 1994).</td>
</tr>
<tr>
<td>PW4b</td>
<td>4 - 6</td>
<td>Recognizes that a sum (whole) in a change add-to problem is greater than the 1) and that a difference (part) is less than the starting amount (whole; Sophian &amp; McCorgray, 1994).</td>
</tr>
<tr>
<td>PW4c</td>
<td>5</td>
<td>Uses qualitative reasoning to determine the direction of an answer of verbally addend (missing-start) change add-to problems (Sophian &amp; McCorgray, 1994).</td>
</tr>
<tr>
<td>PW5</td>
<td></td>
<td>Uses qualitative reasoning to determine the direction of an answer of verbally addend (missing-start) change add-to problems (Sophian &amp; McCorgray, 1994).</td>
</tr>
<tr>
<td>PW5a</td>
<td>5 - 7</td>
<td>Uses qualitative reasoning to determine the direction of an answer of verbally addend (missing-start) change add-to problems (Sophian &amp; McCorgray, 1994).</td>
</tr>
<tr>
<td>PW6</td>
<td>5 - 7</td>
<td>With computational experience, discovers the principle of additive commutativity (Gannon, 1984; Baroody et al., 1983). An understanding of this principle appears in whole situations and only later with change add-to problems (Wilkins et al., 1995).</td>
</tr>
<tr>
<td>PW7</td>
<td>7</td>
<td>Recognizes the effects of transformations to a part or parts on a task involving change add-to problems (Wilkins et al., 1995).</td>
</tr>
<tr>
<td>PW8</td>
<td>~7</td>
<td>Constructs an other-names-for-a-number concept.</td>
</tr>
<tr>
<td>PW9a</td>
<td>7 - 8</td>
<td>Solves verbal part-part-whole problems using quantitative reasoning—Resnick’s (1992) numbers level” (Burghardt &amp; Fuson, 1989; Carpenter &amp; Moser, 1984; Fuson, 1992; Steffon, 1995).</td>
</tr>
</tbody>
</table>

**Note.** PW5b and PW9b would be analogous competencies with numbers. PW9b would be Resnick’s (1992) numbers level. PW5b and PW9b would develop somewhat later than PW5a and PW9a.
PW9a, respectively. It is not clear, though, whether PW5b would develop prior to, simultaneous with, or after PW9a.
Table 2 continued

<table>
<thead>
<tr>
<th>Item</th>
<th>Appx Age</th>
<th>Nonverbal Division &amp; Fraction Concept</th>
</tr>
</thead>
<tbody>
<tr>
<td>NA2</td>
<td></td>
<td>Nonverbal fair sharing</td>
</tr>
<tr>
<td>NA2a</td>
<td>3</td>
<td>• Small collections between two people.</td>
</tr>
<tr>
<td>NA2b</td>
<td>4</td>
<td>• Larger collections between two people.</td>
</tr>
<tr>
<td>NA2c</td>
<td>5</td>
<td>• Collections among three or more people (general dealing-out strategy).</td>
</tr>
<tr>
<td>NA3</td>
<td>4</td>
<td>Appears to solve nonverbal fraction addition and subtraction problems invol</td>
</tr>
</tbody>
</table>

\[
\frac{1}{2} - \frac{1}{4}, \text{ and } 1 - \frac{1}{4} \]

(Mix, Levine, & Huttenlocher, 1999).

Table 3: Major Achievements in the Development of Informal Counting Strategies for Computing Sums and Illustrative Strategies Using 3 + 5 as an Example

<table>
<thead>
<tr>
<th>Achievement</th>
<th>Strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Direct modeling. Children's first appropriate addition strategy typically involves using objects to model the meaning of a problem—to concretely represent and to answer it.</td>
<td>1. Concrete counting-all strategy (CCA)—a counting strategy that involves (a) directly modeling both addends and (b) a separate sum count. For example, for 3 + 5, a child might successively count and raise three fingers to concretely represent the starting amount 3, successively count and raise five fingers to concretely represent the added-on amount 5, and then count all eight fingers to determine the sum.</td>
</tr>
<tr>
<td>2. Short-cutting the direct model. Children take a small, but useful, step by using patterns to short-cut one or more steps in the direct-modeling (CCA) procedure.</td>
<td>2. Concrete counting-all shortcut strategies include using finger patterns to represent each addend. For example, for 3 + 5, a child could automatically (simultaneously) raise three fingers on one hand and then do the same with five fingers on the other hand to represent the addends. The child could then determine the sum by counting all the fingers or by recognizing the number of fingers extended.</td>
</tr>
<tr>
<td>3. Embedded-addend concept = semi-indirect modeling. Children discover that representing an addend (e.g., by counting) can be embedded in (i.e., done simultaneously with) the sum count (Fuson, 1992). As a result, they no longer have to concretely represent an addend. Nevertheless, semi-direct modeling still requires two separate (sequential) counts—one to represent the other addend and a second to determine the sum.</td>
<td>3. Concrete counting of the added-on amount strategies—counting strategies that involve indirectly modeling the starting amount and directly modeling the added-on amount with objects either sequentially or simultaneously before the sum count. For example, for 3 + 5, a child might first extend five fingers to represent the added-on amount, next verbally count up to the cardinal value of the starting amount (one, two, three), and then continue this count as she pointed, in turn, to each of the previously extended fingers (four, five, six, seven, eight). Such strategies are particularly useful for sums greater than 10, making it difficult to represent both addends on the fingers of two hands.</td>
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<tr>
<td>4. Embedded-addends concept + keep- ing track = indirect modeling. Children discover that representing both addends can be done simultaneously with the sum count. This, in turn, entails inventing a keeping-track process (continuing a verbal count from a number for a specified interval). Relatively abstract counting strategies, strategies that do not require separate representations of either addend prior to the sum count are now possible.</td>
<td>4. Abstract counting strategies—counting strategies that involve indirectly modeling both the starting amount and the added-on amount and, thus, require a keeping-track process done in tandem (simultaneously) with a sum count. For 3 + 5, for instance, a child might “count-all beginning with the first addend” (CAF). Verbally count out the starting amount (“One, two, three”) and then count, “four [is one more], five [is two more], six [is three more], seven [is four more], eight [is five more].” Note that the portion in brackets is the keeping-track process.</td>
</tr>
</tbody>
</table>
5. **Disregarding addend order.** Simultaneously executing sum and keeping-track counts puts a heavy burden on working memory. For small-addend-first items such as $3 + 5$, representing the larger addend first can minimize the keeping-track count and, thus, reduce the load on working memory.

6. **Embedded cardinal-count concept = counting-on.** Children discover that, in the context of computing sums, stating the cardinal value of an addend is equivalent to counting up to this number (Fuson, 1992). This permits starting the sum count with the cardinal value of the addend (counting-on) instead of counting from one (counting-all).

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5. **Abstract counting strategies that disregard addend order** such as “counting-all beginning with the larger addend” (CAL). For $3 + 5$, this could take the form of starting with one, counting up to the cardinal value of the larger addend (“...two, three, four, five...”), and then continuing the count for three more terms: “six [is one more], seven [is two more], eight [is three more].”

6. **Abstract counting strategies that involve counting-on** such as “counting-on from the larger addend” (COL). For $3 + 5$, for instance, this might take the form of stating the larger addend (“Five”) and counting on three more times: “six [is one more], seven [is two more], eight [is three more].”

---

*Siegler (e.g., Shrager and Siegler, 1998) refers to CCA as “sum,” and to CAF as “shortcut sum.” Technically, sum refers to the abstract CAF strategy, not to CCA. According to Resnick and Ford (1981, p. 76), the sum model has a reaction time equal to the sum of the two addends ($RT = m + n$), which is approximately the RT a CAF strategy requires. CC requires counting out items to represent each addend and then counting all the items put out, a process that requires a RT equal to twice the sum ($RT = [m] + [n] = m + n = 2(m = n)$.

---

**Figure 3: Models for Grouping and Place-Value Concepts**

Depicted below are increasing abstract models of multidigit numbers using objects or pictures.
### Concrete Models

#### A. Proportional model that requires children to group 10 ones into a ten themselves

![Interlocking blocks](image1)

#### Pictorial Models

![Tally marks](image2)

### Pictorial Models

#### B. Proportional model that involves trading in 10 ones for a pregrouped ten

![Base-ten blocks](image3)

### Concrete Models

#### C. Nonproportional model that involves trading in 10 ones for a different looking ten marker

![Colored chips](image4)

### Pictorial Models

#### D. Nonproportional model that involves trading in 10 ones for an identical marker that represents ten by virtue of its position

![Chalkboard](image5)

### Table 4: Four Approaches to Mathematics Instruction (Baroody, 1998; Baroody et al., in press)

<table>
<thead>
<tr>
<th>Instructional Approach</th>
<th>Philosophical View</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>Skills Approach</td>
<td>Dualism</td>
<td>Right or wrong with no shades of gray: There is one correct procedure or answer.</td>
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<td></td>
<td></td>
<td>Absolute external authority: As the expert, the teacher is the judge of correctness. Pro-</td>
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<table>
<thead>
<tr>
<th>Conceptual Approach</th>
<th>Pluralism</th>
<th>Continuum from right to wrong: There is a choice of possible but not equally valid procedures or answers. Objectively, there is one best possibility.</th>
<th>Tolerant external authority: Teacher accepts diverse procedures and answers but strives for perfection, namely, learning of the best procedure or answer. Teacher provides feedback (e.g., praises all ideas, particularly the conventional one).</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investigative Approach</td>
<td>Instrumentalism</td>
<td>Many right choices: There is a choice of possible procedures or answers and often many are good.</td>
<td>Open internal authority: Teacher or student remains committed to a method or viewpoint as long as it is effective. Teacher responds to incorrect procedures or answers by posing a question, problems, or task that prompts student reflection.</td>
</tr>
<tr>
<td>Problem-Solving Approach</td>
<td>Extreme Relativism</td>
<td>No right or wrong: There are many possible, equally valid possibilities.</td>
<td>No external authority: Teacher and each student define his or her own truth. Children evaluate their own conclusions</td>
</tr>
</tbody>
</table>

(teachering by imposition)

Semi-authoritarian and teacher centered: Direct and semi-direct instruction (teachering by “careful imposition”)

Semi-democratic and student-centered: Semi-indirect instruction (guided participatory democracy)

Completely democratic and extremely student-centered: Indirect instruction (teaching by negotiation)
### Table 4 continued

<table>
<thead>
<tr>
<th>Focus of Instruction</th>
<th>Teacher's/Students' Roles</th>
<th>Organizing Principle</th>
<th>Methods</th>
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</thead>
<tbody>
<tr>
<td>Skills Approach</td>
<td>Procedural content (e.g., how to add multidigit numbers)</td>
<td>Teacher serves as a director: an</td>
<td>Bottom-up (logically): Sequential instruction from most basic skills to most complex skills as problem solving,</td>
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<td>information dispenser (informer) and</td>
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<td>taskmaster (manager). Because</td>
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<td>children are viewed as uninformed</td>
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<td>and helpless, students must be</td>
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<td>spoonfed knowledge.</td>
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<tr>
<td>Conceptual Approach</td>
<td>Procedural and conceptual content (e.g., why you carry when adding multidigit numbers)</td>
<td>Teacher serves as a shepherd:</td>
<td>Bottom-up (psychologically): Sequential instruction based on the readiness of students to construct understanding,</td>
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<td>information dispenser (informer) and</td>
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<td>up-front guide (conductor). Because</td>
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<td>children are seen as capable of</td>
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<td>understanding mathematics if helped,</td>
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<td>they are engaged in quasi-</td>
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<td>independent activities and</td>
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<td>discussions.</td>
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<tr>
<td>Investigative Approach</td>
<td>Procedural content, conceptual content, and the processes of mathematical inquiry</td>
<td>Teacher serves as a mentor:</td>
<td>Top-down (guided): Teacher usually poses a &quot;worthwhile task&quot; (one that is challenging and complex) as way of exploring,</td>
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<td></td>
<td>(problem solving, reasoning, conjecturing, representing and communicating)</td>
<td>activity organizer (instigator) and</td>
<td>learning and practicing basic concepts and skills; teacher may take</td>
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<td>guide-on-the-side (moderator).</td>
<td>advantage of teachable moments (e.g., question or problem posed by</td>
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<td>Because children have informal</td>
<td>student).</td>
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<td>knowledge and an inherent need to</td>
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<td>understand, they are capable of</td>
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<td>inventing their own solutions and</td>
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<td>making (at least some) sense of</td>
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<td>mathematical situations themselves</td>
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<td>(i.e., students are engaged in</td>
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<td>semi-independent activities and</td>
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<td>discussions).</td>
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<tr>
<td>Problem-Solving Approach</td>
<td>Processes of mathematical inquiry: problem solving, reasoning, conjecturing representing</td>
<td>Teacher serves as a partner:</td>
<td>Top-down (unguided): Class tackles problems of their own choosing,</td>
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<td>participant, monitor, and devil’s</td>
<td>whether or not students have received formal</td>
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<td>advocate. Students engage in</td>
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<td>relatively independent</td>
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Goals

N1. Instruction from prekindergarten to grade 2 should enable all students to understand the roles of numbers, relations among them, and both informal and formal ways of representing numbers relations (based on Really Big Idea 1).

N1.1 Use numbers to count with understanding—that is, connect number words to the quantities they represent so as to recognize how many in a collection or to count out collections of a particular size (based on Big Idea 1).

N1.1.1 Nonverbally represent small collections of 1 to 4 items (Item NN1 in Table 2).
  N1.1.1a Nonverbally subitize and produce one or two items (NN1a).
  N1.1.1b Nonverbally identify as equivalent or nonequivalent static (simultaneously presented) collections consisting of 1 to 4 items (NN1b).
  N1.1.1c Nonverbally produce up to 4 items (NN1c).
  N1.1.1d Nonverbally match equivalent sets of disks and dots—two homogeneous sets consisting of highly similar items (NN1d).

N1.1.2 Use (verbal or nonverbal) subitizing or (covert or overt) counting to make nonverbal equivalence judgments of small collections (NN2).

N1.1.2a Nonverbally match equivalent sets of shells and dots—two homogeneous sets consisting of dissimilar items (NN2a).

N1.1.2b Nonverbally identify a (static) collection (e.g., 3 dots) with a sequential presentation (e.g., 3 successively presented dots; NN2b).

N1.1.2c Nonverbally make a cross-modal match (e.g., match 3 dots with 3 bell rings; NN2c).

N1.1.2d Nonverbally match a static collection with a sequentially repeated event (e.g., match 3 dots with 3 puppet jumps; NN2d).

N1.1.2e Nonverbally match equivalent sets of random objects and dots—a heterogeneous collection and a set of dissimilar items (NN2e).

N1.1.3 String number words together to create the counting (by ones) sequence (VC1).

N1.1.3a Use a nonstandard sequence, such as “two five” or “twothreeeten” (VC1a).

N1.1.3b Start sequence with one, e.g., “one two” (VC1b).

N1.1.3c Cite numbers in the correct sequence to “ten” (VC2b).

N1.1.3d Use teen pattern to cite correct sequence to “nineteen.”

Figure 4 continued

Goals

N1.1.3e Use repeating patterns to cite correct sequence to “twenty-nine.”

N1.1.3f Use repeating patterns to cite correct sequence to “one hundred.”
N1.1.4 Use the counting sequence to enumerate collections—that is, to count objects to identify the number of items in a collection (VC3).

N1.1.4a Verbally subitizes (recognizes and identifies by number) one or two items (VC3a).

N1.1.4b Pair a number word of the count sequence with each item of a small collection and identify a collection by repeating the count (VC3b).

N1.1.4c Verbally subitizes up to four or five items (VC3c).

N1.1.4d Construct the count-cardinal concept—that is, recognize that the last number word used to label items in a collection also represents the total (VC3d).

N1.1.4e Accurately enumerate any type of collection of up to five items (VC3e).

N1.1.4f Recognize identity-conservation or number-constancy principle (VC3f).

N1.1.4g Recognize the order-irrelevance principle (VC3g).

N1.1.4h Accurately enumerate collections to 10.

N1.1.4i Accurately enumerate collections to 20.

N1.1.5 Use the counting sequence to produce (count out) collections of a specified size.

N1.1.5a Quasi-verbal production of one or two items (e.g., responds to a verbal request for two items by subitizing and offering two items or by holding up two fingers; VC4a).

N1.1.5b Quasi-verbal production of up to four or five items (e.g., responds to a verbal request for four items by subitizing and offering four items or by holding up four fingers; VC4b).

N1.1.5c Construct the cardinal-count concept—that is, recognize that a cardinal label for a collection such as “four” is equivalent to actually counting the collection (VC4c).

N1.1.5d Accurately produce up to 5 items in response to a verbal request (verbal production; VC4d).

N1.1.5e Verbal production of up to 10 items.

N1.1.5f Verbal production of up to 20 items.

N1.1.6 Flexibly start verbal count-by-one sequence from any point—that is, start a count from a number other than “one” (VC5a).

N1.1.7 Flexibly cite the number after a specified count term.

N1.1.7a State number after 1 to 9 with a running start (e.g., “What comes after 1, 2, 3, 4, 5?”).

N1.1.7b State number after 1 to 9 with an abbreviated running start (e.g., “What comes after 3, 4, 5?”).

N1.1.7c State number after 1 to 9 without any running start (VC5a).

Figure 4 continued

Goals

N1.1.8a State number before 2 to 10.

N1.1.8b State number before 11 to 29.

N1.1.9 Verbally count backward.

N1.1.9a Verbally count backward form “five.”

N1.1.9b Verbally count backward from “ten” (VC5b).

N1.1.9c Verbally count backward form “twenty.”

N1.1.10 Apply decade-count skills.

N1.1.10a Skip count by tens to 100.

N1.1.10b Flexibly state the decade after 10 to 90.

N1.1.11 Flexibly use other skip counts (VC5c).

N1.1.11a Verbally count by fives to 100.

N1.1.11b Verbally count by twos to 20.
N1.1.11c Count objects by fives.
N1.1.11d Count objects by twos.
N1.1.11e Verbally count odd numbers to 19.

N1.12 Explicitly distinguish between the cardinal meaning/use of number and other (ordinal, measurement, and nominal) meanings/uses of number.

N1.2 Use numbers to compare quantities by developing an understanding of the relative position and magnitude of whole numbers and the connection between ordinal and cardinal numbers.

N1.2.1 Visually identify whether collections are the “same” (number) or which is “more” (VN1).

N1.2.1a Correctly indicate “same” or “more” with collections that are obviously equal or not equal (perception of “same” or “more”).

N1.2.1b Correctly indicate “more” with collections up to about 4 that differ in number by one (perception of fine differences).

N1.2.2 Recognize equivalent collections despite appearances (VN2).

N1.2.2a Recognize equivalent collections despite appearances. Apply the same-number (number-identity) principle: Two collections with same cardinal designation are equal in number regardless of appearances (VN2a).

N1.2.2b Conserves number (VN2b).

N1.2.3 Use larger-number principle (the later a number appears in the counting sequence, the larger the quantity represented) to make gross comparisons—that is, to compare widely separated numbers (VN3).

N1.2.3a Make gross comparisons of “more” up to “ten” (VN3a).

N1.2.3b Make gross comparisons of “less” up to “ten.”

Figure 4 continued

Goals

N1.2.3c Make gross comparison of “more” up to “one hundred.”

N1.2.4 Use larger number principle and number after knowledge to make fine comparisons—that is, to compare two adjacent numbers in the counting sequence (VN4).

N1.2.4a Make fine comparisons of more up to “five” (VN4a).

N1.2.4b Make fine comparisons of more up to “ten” (VN4b).

N1.2.4c Make fine comparisons of more up to “one hundred.”

N1.2.4d Make fine comparisons of less up to “ten.”

N1.2.4e Make fine comparisons of less up to “one hundred.”

N1.2.5 Understand and effectively apply verbal ordinal terms.

N1.2.5a Recite the ordinal terms (first, second, third…) to “tenth.”

N1.2.5b Describe the parallels and differences between the ordinal and cardinal sequences.

N1.2.5c Recognize that ordinal terms are relational—are meaningful only if a point of reference is specified.

N1.2.5d Recite and apply effectively ordinal terms to “twenty-ninth.”

N1.3 Represent collections up to 10 and numerical relations by connecting numerals to number words and the quantities both represent.

N1.3.1 Draw pictographic symbols (drawings of objects) or iconic symbols (e.g., tallies) to response to a spoken number (nonfunctional use of informal numerical symbols).

N1.3.2 Use pictographic or iconic symbols to represent the cardinal value of a collection (functional use of informal numerical symbols).

N1.3.3 Execute and apply numeral skills.

N1.3.3a Recognize/identify one-digit numerals (e.g., is able to point out a “three” given a choice of fine numerals; WN1a).

N1.3.3b Read one-digit numerals (WN1b).

N1.3.3c Copy or write one-digit numerals (WN2a & WN2b).

N1.3.3d Use one-digit written numbers to represent the cardinal value of a collection (functional use of numerals).
N1.3.4 Use relational symbols effectively.
N1.3.4a Informally represent the equivalence or inequivalence of two collections.
N1.3.4b Correctly identify and use the formal relational symbols =, ≠, >, < with single-digit numbers.

N1.3.5 Use written number words and relational terms effectively.
N1.3.5a Identify written number words one, two, three, . . . nine with their corresponding verbal words and numerals and use them to represent the cardinal value of a collection.

Figure 4 continued

Goals
N1.3.5b Describe the parallels between abbreviated ordinal terms (1st, 2nd, 3rd . . . 9th) and cardinal terms.
N1.3.5c Identify written ordinal terms first, second, third, . . . ninth with their corresponding verbal words and use them to represent ordinal relations.
N1.3.5d Identify written relational terms equals, unequal, greater than, and less than with their corresponding verbal terms and written symbols.

O2. Instruction from pre-kindergarten to grade 2 should enable all students to understand the various meanings of operations, to recognize how the operations are related, to compute fluently, and to make reasonable estimates (Really Big Idea 2).

O2.1 Understand the change meaning of addition and subtraction of whole numbers (Big Idea 1) and use this knowledge to make sensible estimates and to develop calculational proficiency.

O2.1.1 Nonverbally and mentally determine sums and differences.
O2.1.1a Nonverbally add one item and another or subtract one item from two (NA1a).
O2.1.1b Nonverbally estimate sums up to five and their subtraction complements (e.g., for “3 + 2” put out 4 to 6 items as the answer).
O2.1.1c Nonverbally determine sums up to three and differences up to “3 – 2” (NA1b).
O2.1.1d Nonverbally determine sums up to five and their subtraction counterparts (NA1c).

O2.1.2 Estimate the sums of addition word problems and their subtraction complements up to . . .
O2.1.2a 10;
O2.1.2b 20.

O2.1.3 Use direct-modeling strategies (concrete counting-all or take-away) to solve addition and subtraction word problems (VA1a) with . . .
O2.1.3a Sums to 10 and corresponding differences;
O2.1.3b Sums to 18 and corresponding differences.

O2.1.4 Use more advanced counting strategies to solve addition word problems with sums to 18.
O2.1.4a Use the embedded-addend concept to indirectly model addition (i.e., use verbal counting-all; VA2).
O2.1.4b Use the number-after rule to determine sums for n + 1 and 1 + n combinations
O2.1.4c Use the embedded cardinal-count concept to solve addition problems by counting-on and subtraction problems by counting-down or –up.

Figure 4 continued

Goals
O2.1.5 Connect formal addition and subtraction to concrete or informal knowledge.
O2.1.5a Translate addition and subtraction word problems (and their solutions) into a number sentence and vice versa (VA1b).

O2.1.5b Solve symbolic expression using a variety of strategies.

O2.1.6 Use thinking strategies and existing knowledge to reason out unknown sums to 18 and their subtraction counterparts (e.g., $7 + 8 = 7 + 7 + 1 = 14 + 1 = 15$ or $15 - 7 = ? \rightarrow 7 + ? = 15 \rightarrow$ so $? = 8$).

O2.1.7 Achieve fluency with basic addition and subtraction combinations regardless of strategy used.